Cosmological perturbations in the models of dark energy and modified gravity

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J. Matsumoto, Phys. Rev. D **87**, 104002 (2013)
K. Bamba, J. Matsumoto, S. Nojiri, Phys. Rev. D **85**, 084026 (2012)
J. Matsumoto, Phys. Rev. D **83**, 124040 (2011)

Introduction I

Observations of type Ia supernovae give us the information about background evolution of the Universe.

↙ ↘

Homogeneous Universe

↓

Accelerated expansion of the Universe

Inhomogeneous Universe

→ Modification of the Einstein equation is needed.

$$R_{\mu\nu} - g_{\mu\nu}R/2 = \kappa^2 T_{\mu\nu}$$

Modified Gravity

F(R) gravity
Scalar-Tensor theory
Galileon
…

Dark Energy

Cosmological Constant
Quintessence
Ghost Condensate
…

(Quintessence, Ghost Condensate) → *k*-essence

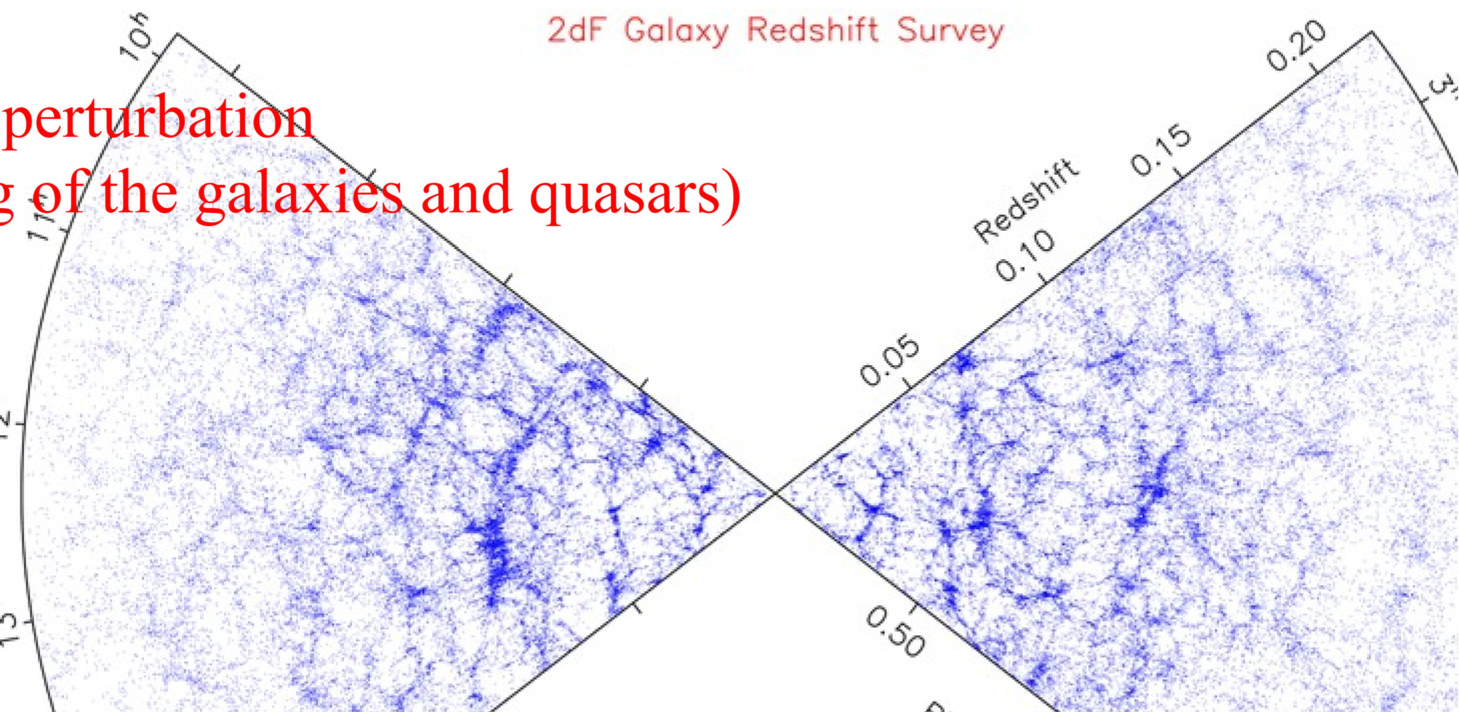
Introduction II

How to clarify which model describes the nature?

- Background evolution of the Universe (SN Ia, CMB, BAO, ...)

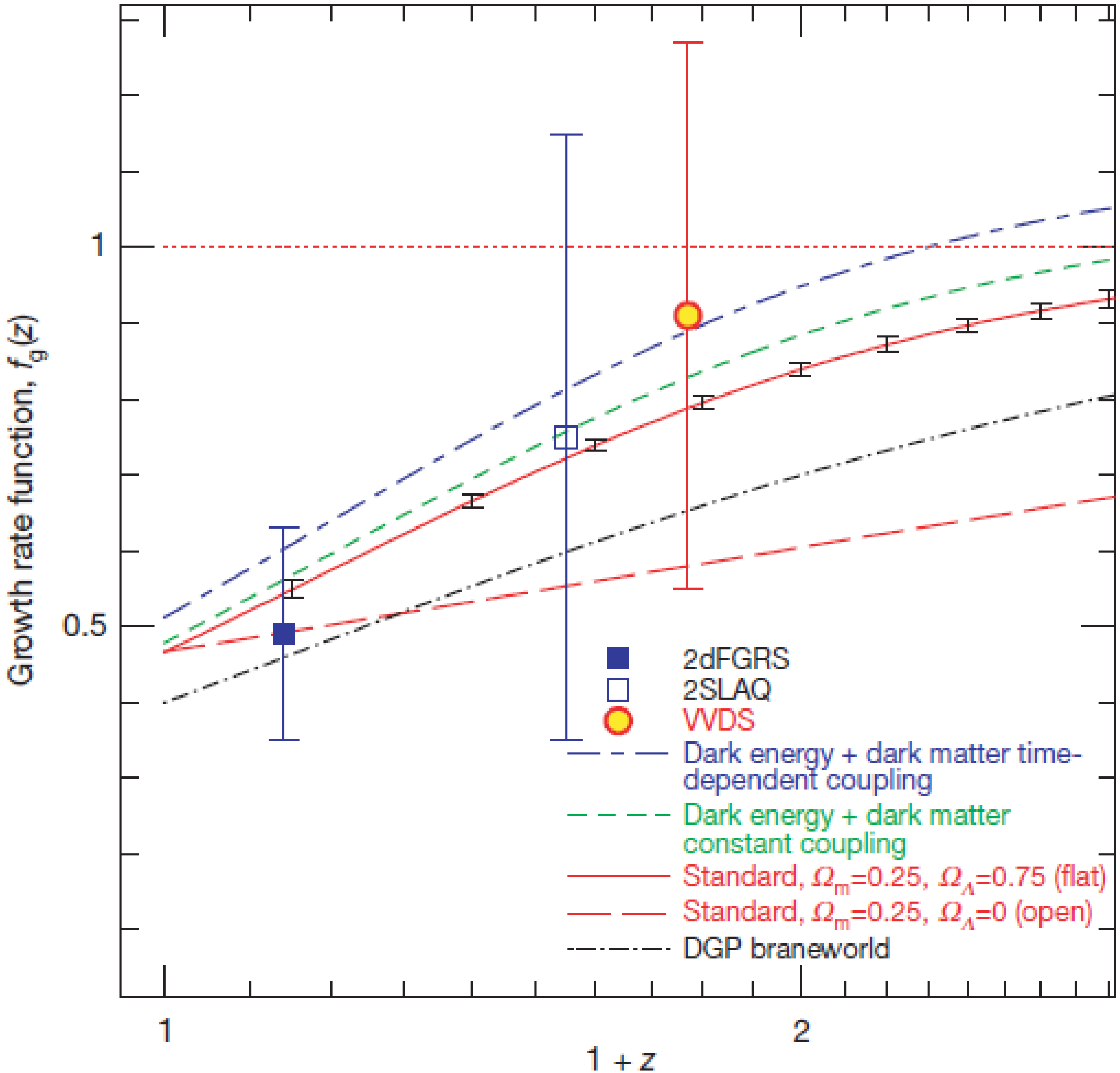
However, the formalism, `` reconstruction,” have been developed in dark energy and modified gravity models. Therefore, we can not find the differences between models in the background evolution of the Universe.

- The matter density perturbation
  (LSS, Microlensing of the galaxies and quasars)



Introduction III

L. Guzzo et al., Nature, 451, 541 (2008)



Growth rate

$f = d\ln\delta / dN$

Matter density perturbation $\delta = \delta\rho/\rho$

$N = \ln a = - \ln (1+z)$

The ΛCDM model

FL equations in the ΛCDM model

Einstein equation: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -g_{\mu\nu}\Lambda + \kappa^2 T_{\mu\nu}.$ $\kappa^2 = 8\pi G$

Metric: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j.$

Friedmann-Lemaitre equations

$$3H^2 = \kappa^2\rho + \Lambda,$$

$$-a^2\delta_{ij}(2\dot{H} + 3H^2) = \kappa^2 a^2 \delta_{ij} p - a^2\delta_{ij}\Lambda,$$

ρ: Energy density
p: Pressure

$$H(t) \equiv \dot{a}(t)/a(t).$$

The equation of continuity

$$\rho + 3(1+w)H\rho = 0.$$

Equation of state parameter

$$w = p/\rho$$

Perturbative equations

$$\delta R^{\nu}{}_{\mu} - \delta^{\nu}{}_{\mu}\frac{1}{2}\delta R = \kappa^2 \delta T^{\nu}{}_{\mu},$$

Newtonian gauge $ds^2 = (-1+2\Phi)dt^2 + \delta_{ij}a(t)^2(1+2\Psi)dx^i dx^j,$

Einstein equations

$$\delta T^0{}_0 = -\delta\rho\,,$$

$$\delta T^0{}_i = (\rho+p)\delta u_i\,,$$

$$\delta T^i{}_0 = -a^{-2}(\rho+p)\delta u_i\,,$$

$$\delta T^i{}_j = \delta^i{}_j \delta p\,,$$

(0,0) $$-6H^2\Phi - 2\frac{k^2}{a^2}\Psi - 6H\partial_0\Psi = -\kappa^2\delta\rho,$$

(0,i) $$2\partial_i(H\Phi + \partial_0\Psi) = \kappa^2(\rho+p)\delta u_i,$$

(i,j) $$a^{-2}\partial_i\partial_j(\Phi-\Psi) = 0, \quad (i\neq j),$$

(i,i) $$\left(\frac{k^2}{a^2} + \frac{\partial_i\partial_i}{a^2} - 2H\partial_0 - 4\dot{H} - 6H^2\right)\Phi - \left(\frac{k^2}{a^2} + \frac{\partial_i\partial_i}{a^2} + 2\partial_0\partial_0 + 6H\partial_0\right)\Psi = \kappa^2\delta p, \quad (\text{not summed with respect to } i)\,,$$

The equations of continuity

(0) $$\delta\dot{\rho} + 3H(\delta\rho+\delta p) + a^{-2}\partial_i\{(\rho+p)\delta u_i\} + 3\dot{\Psi}(\rho+p) = 0,$$

$\delta = \delta\rho/\rho$

(i) $$a^{-3}\partial_0\{a^3(\rho+p)\delta u_i\} + \partial_i\delta p - (\rho+p)\partial_i\Phi = 0.$$

$c_s^2 \equiv \delta p/\delta\rho$

Subhorizon approximation

Quasi-static approximation

$$\dot{\phi} \sim H\phi, \dot{\psi} \sim H\psi, \dot{\delta} \sim H\delta, \dot{H} \sim H^2.$$

Small scale approximation

a/k << 1/H

e.g. Virgo Supercluster:
a/k = 33 Mpc

Einstein equations

(0,0) $$-6H^2\Phi - 2\frac{k^2}{a^2}\Psi - 6H\partial_0\Psi = -\kappa^2\delta\rho,$$

(0,i) $$2\partial_i(H\Phi + \partial_0\Psi) = \kappa^2(\rho + p)\delta u_i,$$

(i,j) $$a^{-2}\partial_i\partial_j(\Phi - \Psi) = 0, \quad (i \neq j),$$

(i,i) $$\left(\frac{k^2}{a^2} + \frac{\partial_i\partial_i}{a^2} - 2H\partial_0 - 4\dot{H} - 6H^2\right)\Phi - \left(\frac{k^2}{a^2} + \frac{\partial_i\partial_i}{a^2} + 2\partial_0\partial_0 + 6H\partial_0\right)\Psi = \kappa^2\delta p, \quad (\text{not summed with respect to } i),$$

The equations of continuity

(0) $$\delta\dot{\rho} + 3H(\delta\rho + \delta p) + a^{-2}\partial_i\{(\rho + p)\delta u_i\} + 3\dot{\Psi}(\rho + p) = 0,$$

$\delta = \delta\rho/\rho$

(i) $$a^{-3}\partial_0\{a^3(\rho + p)\delta u_i\} + \partial_i\delta p - (\rho + p)\partial_i\Phi = 0.$$

$c_s^2 \equiv \delta p/\delta\rho$

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2 \delta \simeq 0,$$

↕ Consistent

~~Quasi-static approximation~~

~~$\dot{\phi} \sim H\phi, \dot{\psi} \sim H\psi, \dot{\delta} \sim H\delta, \dot{H} \sim H^2.$~~

Small scale approximation

H << k/a

$$\ddot{\delta} + \left\{2 - 3w + O\left(\left(\frac{k^2}{a^2H^2}\right)^{-1}\right)\right\} H\dot{\delta} + \left\{\frac{c_s^2 k^2}{a^2H^2} - \frac{3}{2}(1+w)(1+3w)\Omega_m + O\left(\left(\frac{k^2}{a^2H^2}\right)^{-1}\right)\right\}\delta = 0.$$

Growth rate

f = dlnδ / dN = 1 in matter dominant era.

k-essence model

k-essence model

$$S = \int d^4x \sqrt{-g} \left\{ \frac{R}{2\kappa^2} - K(\phi, X) + L_{\rm matter} \right\}, \quad X \equiv -\frac{1}{2}\partial^\mu \phi \partial_\mu \phi \,.$$

Metric: $ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$.

FL equations

$$\frac{3H^2}{\kappa^2} = K - \dot{\phi}^2 K_{,X} + \rho_{\rm matter} \,,$$

$$-\frac{2\dot{H}}{\kappa^2} = -\dot{\phi}^2 K_{,X} + (1+w)\rho_{\rm matter} \,,$$

$K_{,X} = \partial K/\partial X$,
$K_{,\phi} = \partial K/\partial \phi$.

EoM: $0 = 3H\dot{\phi}K_{,X} + \ddot{\phi}(K_{,X} + \dot{\phi}^2 K_{,XX}) - K_{,\phi} + \dot{\phi}^2 K_{,X\phi} \,,$

The equation of continuity

$$\dot{\rho}_{\rm matter} + 3(1+w)H\rho_{\rm matter} = 0 \,.$$

$w = p/\rho$

The differential equation of the matter density perturbation

$$\frac{d^4\delta}{dN^4}+M_3\frac{d^3\delta}{dN^3}+M_2\frac{d^2\delta}{dN^2}+M_1\frac{d\delta}{dN}+M_0\delta=0$$

$$N=\ln a$$

$$\begin{aligned}M_3=&\Bigg[7+3c_s^2-6w-\frac{2\ddot{\phi}}{H\dot{\phi}}-3(1+w)\frac{\kappa^2\rho}{H^2}+\frac{3\kappa^2\dot{\phi}^2K_{,X}}{H^2}-\frac{6K_{,X}}{4K_{,X}+\dot{\phi}^2K_{,XX}}\left(3+\frac{\dddot{\phi}}{H\ddot{\phi}}\right)\\&-\frac{\dot{\phi}(10K_{,X}+\dot{\phi}^2K_{,XX})(\ddot{\phi}K_{,XX}+K_{,\phi X})}{HK_{,X}(4K_{,X}+\dot{\phi}^2K_{,XX})}+\frac{3\dot{\phi}K_{,X}}{H\ddot{\phi}(K_{,X}+\dot{\phi}^2K_{,XX})(4K_{,X}+\dot{\phi}^2K_{,XX})}\\&\times\left\{3(1+w)\kappa^2\rho K_{,X}-3\kappa^2\dot{\phi}^2K_{,X}^2+2K_{,\phi\phi}-2\ddot{\phi}K_{,\phi X}-6H\dot{\phi}K_{,\phi X}-2\dot{\phi}^2K_{,\phi\phi X}-2\dot{\phi}^2\ddot{\phi}K_{,\phi XX}\right\}\Bigg]+O\left(\frac{a^2H^2}{k^2}\right),\end{aligned}$$

$$M_2=\frac{k^2}{a^2H^2}\left(c_s^2+\frac{K_{,X}}{K_{,X}+\dot{\phi}^2K_{,XX}}\right)+O(1),$$

$$\begin{aligned}M_1=&\frac{k^2}{a^2H^2}\Bigg[c_s^2\left\{-\frac{(1+w)\kappa^2\rho}{2H^2}+\frac{\kappa^2\dot{\phi}^2K_{,X}}{2H^2}-\frac{2\ddot{\phi}}{H\dot{\phi}}\right\}\\&+\frac{K_{,X}}{K_{,X}+\dot{\phi}^2K_{,XX}}\left\{2-6w-\frac{(1+w)\kappa^2\rho}{2H^2}+\frac{\kappa^2\dot{\phi}^2K_{,X}}{2H^2}\right\}\\&+c_s^2\frac{-2K_{,X}+\dot{\phi}^2K_{,XX}-6\frac{\dddot{\phi}}{H\ddot{\phi}}K_{,X}}{4K_{,X}+\dot{\phi}^2K_{,XX}}-c_s^2\frac{\dot{\phi}(10K_{,X}+\dot{\phi}^2K_{,XX})(\ddot{\phi}K_{,XX}+K_{,\phi X})}{HK_{,X}(4K_{,X}+\dot{\phi}^2K_{,XX})}\\&+c_s^2\frac{3K_{,X}}{H\ddot{\phi}(K_{,X}+\dot{\phi}^2K_{,XX})(4K_{,X}+\dot{\phi}^2K_{,XX})}\Big\{-3H\dot{\phi}^2(\ddot{\phi}K_{,XX}+2K_{,\phi X})\\&+3(1+w)\kappa^2\rho\dot{\phi}K_{,X}-3\kappa^2\dot{\phi}^3K_{,X}^2-2\dot{\phi}\ddot{\phi}K_{,\phi X}+2\dot{\phi}K_{,\phi\phi}\end{aligned}$$

$$M_0=\frac{k^4}{a^4H^4}\left(\frac{c_s^2K_{,X}}{K_{,X}+\dot{\phi}^2K_{,XX}}\right)+O\left(\frac{k^2}{a^2H^2}\right).$$

WKB approximated solutions

Small scale approximation H << k/a

$c_s^2 = w = 0$

Quasi-static solutions

$$c_\phi^2 \left\{ \frac{d^2\delta}{dN^2} + \left(\frac{1}{2} - \frac{3}{2}w_{\rm eff}\right)\frac{d\delta}{dN} - \frac{3}{2}\Omega_{\rm m}\delta \right\} = 0.$$

$$N = \ln a$$

Oscillating solutions

$$C_3(N)\cos\left[\int^N dN' \frac{c_\phi k}{aH}\right] + C_4(N)\sin\left[\int^N dN' \frac{c_\phi k}{aH}\right]$$

$$\begin{aligned}\frac{d}{dN}\ln|C(N)| =& \frac{d}{dN}\ln|\dot\phi| - \frac{3}{4}\frac{d}{dN}\ln|K_{,X}| \\ &+ \frac{1}{4}\frac{d}{dN}\ln\left|K_{,X} + \dot\phi^2 K_{,XX}\right| + \frac{d}{dN}\ln|4K_{,X} + \dot\phi^2 K_{,XX}|\end{aligned}$$

Sound speed in *k*-essence model: $c_\phi^2 \equiv (p_\phi)_{,X}/(\rho_\phi)_{,X} = K_{,X}/(K_{,X} + \dot\phi^2 K_{,XX})$

Quasi-static solutions are same as those in the ΛCDM model at leading order.

An example

Exponential potential quintessence model

C. Rubano and P. Scudellaro, Gen. Rel. Grav. **34**, 307 (2002)

$K(\phi,X) = -X + B^2 e^{-\sqrt{3/2}\kappa\phi}$

Effective growth factor

$$f_{\rm eff} \equiv \frac{d}{dN}\ln|C_3(N)| = \frac{d}{dN}\ln|C_4(N)| = \frac{\ddot{\phi}}{H\dot{\phi}} \,.$$

$$a^3(t) = (u_1 t + u_2)\left(\frac{1}{4}u_1\kappa^2 B^2 t^3 + \frac{3}{4}u_2\kappa^2 B^2 t^2 + v_1 t + v_2\right) \,,$$

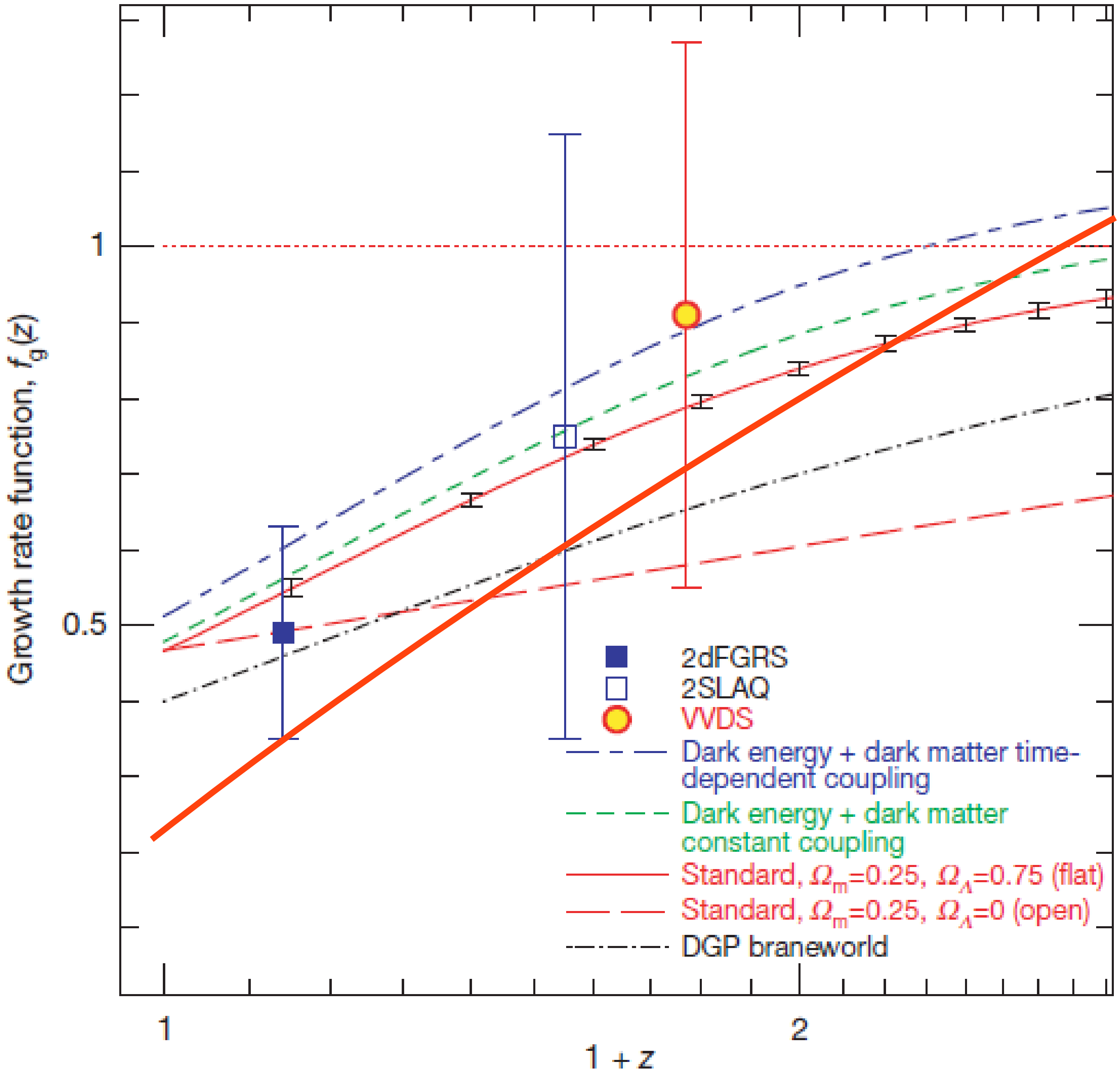
$$\phi(t) = -\frac{\sqrt{2}}{\sqrt{3}\kappa}\ln\frac{u_1 t + u_2}{\frac{1}{4}u_1\kappa^2 B^2 t^3 + \frac{3}{4}u_2\kappa^2 B^2 t^2 + v_1 t + v_2} \,,$$

$\lim_{t\to 0} f_{\rm eff} = 3/2$ $\lim_{t\to\infty} f_{\rm eff} = -3/4$

$-3/4 \le f_{\rm eff} \le 3/2$ $f_{\rm eff}(t_0) \approx 0.21$

Oscillating solutions are dominant mode in this model.

L. Guzzo et al., Nature, 451, 541 (2008)



Growth rate

$f = d\ln\delta / dN$

Matter density perturbation $\delta = \delta\rho/\rho$

$N = \ln a = - \ln (1+z)$

F(R) gravity

FL equations in F(R) gravity

Action: $S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,[R + f(R)] + S_{\rm matter}.$ $\kappa^2 = 8\pi G$

Metric: $ds^2 = -a^2(\eta)(-d\eta^2 + \delta_{ij}dx^i dx^j).$

Friedmann-Lemaitre equations

$$\frac{3\mathcal{H}'}{a^2}(1 + f_R) - \frac{1}{2}(R + f) - \frac{3\mathcal{H}}{a^2}f'_R = -\kappa^2\rho$$

$$\frac{1}{a^2}(\mathcal{H}' + 2\mathcal{H}^2)(1 + f_R) - \frac{1}{2}(R + f) - \frac{1}{a^2}(\mathcal{H}f'_R + f''_R) = \kappa^2 w\rho$$

$w = p/\rho$ $\mathcal{H} \equiv a'/a$ $R = 6a^{-2}(\mathcal{H}' + \mathcal{H}^2)$ $f_R \equiv df(R)/dR$

The equation of continuity

$$\rho' + 3(1 + w)\mathcal{H}\rho = 0.$$

Viable models of F(R) gravity

W. Hu and I. Sawicki, Phys. Rev. D **76**,064004 (2007).

$$f(R) = -m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1}$$

A. A. Starobinsky, JTEP Lett. **86**, 157 (2007).

$$f(R) = \lambda R_0 \left(\left(1 + \frac{R^2}{R_0^2} \right)^{-n} - 1 \right)$$

E. V. Linder, Phys. Rev. D **80**, 123528 (2009); G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini, Phys. Rev. D **77**, 046009 (2008).

$$f(R) = -cr(1 - \mathrm{e}^{-R/r}),$$

The parameters in these models are tuned to be $|f_R| \ll 1$ to satisfy local gravity constraints, and $|f_{RR}|k^2/a^2 \ll 1$ would be satisfied as a result of it.

Best fit parameters

Best fit models to the Sloan Digital Sky Survey (SDSS) and the seven years data of the Wilkinson Microwave Anisotropy Prove (WMAP7)

V. F. Cardone, S. Camera, and A. Diaferio, [arXiv:1201.3272 [astro-ph.CO]]

W. Hu and I. Sawicki, Phys. Rev. D **76**,064004 (2007).

$n = 1.53$, $c_1 = 10^{3.47}$ and $c_2 = 10^{2.28}$, $\quad f(R) = -m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1}$

A. A. Starobinsky, JTEP Lett. **86**, 157 (2007).

$n = 1.34$, $\lambda = 10^{1.50}$ and $R_\star/R_0 = 10^{-1.73}$, $\quad f(R) = \lambda R_\star \left(\left(1 + \frac{R^2}{R_\star^2} \right)^{-n} - 1 \right)$

The fourth order differential equation

$|f_R|k^2/H^2,\ |f_{RR}|k^2/a^2 << 1$

$$\delta'''' + \left\{ \frac{12\mathcal{H}^2(-2+\mathcal{H}''/\mathcal{H}^3)f_{RRR}}{a^2 f_{RR}} + \frac{1-\mathcal{H}'/\mathcal{H}^2}{-2+\mathcal{H}''/\mathcal{H}^3} + O(\mathcal{H}^2/\chi^2) \right\} \mathcal{H}\delta'''$$
$$+\chi^2 \left\{ \left(1+O(\mathcal{H}^2/\chi^2)\right)\delta'' + \mathcal{H}\left(1+O(\mathcal{H}^2/\chi^2)\right)\delta' + \mathcal{H}^2\left(2\frac{\mathcal{H}'}{\mathcal{H}^2} - \frac{\mathcal{H}''}{\mathcal{H}^3} + O(\mathcal{H}^2/\chi^2)\right)\delta \right\} = 0,$$

$$\chi \equiv \sqrt{\frac{a^2}{3f_{RR}}\left(\frac{1-\mathcal{H}'/\mathcal{H}^2}{-2+\mathcal{H}''/\mathcal{H}^3}\right)}.$$

Two decomposed equations

$\partial/\partial\eta \sim \mathcal{H}$, $\mathcal{H} << k, \chi$

$$\longrightarrow \quad \frac{d^2\delta}{dN^2} + \left(\frac{1}{2} - \frac{3}{2} w_{\rm eff}\right)\frac{d\delta}{dN} + \left(2\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3}\right)\delta = 0.$$

$$2\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3} = -\frac{3\Omega_{\rm m}}{2(1+f_R)} - \frac{f_R'}{\mathcal{H}(1+f_R)} - \frac{f_R''}{\mathcal{H}^2(1+f_R)} + \frac{f_R'''}{2\mathcal{H}^3(1+f_R)}.$$

$\partial/\partial\eta \sim \chi$, $\mathcal{H} << k, \chi$

$$\longrightarrow \quad \delta(\eta) = C_1 \mathrm{e}^{\int f_{\rm eff} dN + i\int \chi d\eta} + C_2 \mathrm{e}^{\int f_{\rm eff} dN - i\int \chi d\eta}.$$

$$f_{\rm eff} = 1 - \frac{5}{2}\frac{d}{dN}\ln|\chi| - 2\frac{d}{dN}\ln|f_{RR}| + \frac{1 - \mathcal{H}'/\mathcal{H}^2}{2 - \mathcal{H}''/\mathcal{H}^3}.$$

Behaviors of the solutions I

Quasi-static modes

$$\frac{d^2\delta}{dN^2} + \left(\frac{1}{2} - \frac{3}{2}w_{\rm eff}\right)\frac{d\delta}{dN} + \left(2\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3}\right)\delta = 0.$$

$N = \ln a$

$$2\frac{\dot{H}}{H^2} + \frac{\ddot{H}}{H^3} = -\frac{3\Omega_{\rm m}}{2(1+f_R)} - \frac{f_R'}{\mathcal{H}(1+f_R)} - \frac{f_R''}{\mathcal{H}^2(1+f_R)} + \frac{f_R'''}{2\mathcal{H}^3(1+f_R)}.$$

Negligible

$\longrightarrow$ $\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_{\rm m}H^2\delta \approx 0$

Same as ΛCDM model's

f = dlnδ/dN = 1

Behaviors of the solutions II

$$\delta(\eta) = C_1 \mathrm{e}^{\int f_{\rm eff} dN + i\int \chi d\eta} + C_2 \mathrm{e}^{\int f_{\rm eff} dN - i\int \chi d\eta}.$$

$$f_{\rm eff} = 1 - \frac{5}{2}\frac{d}{dN}\ln|\chi| - 2\frac{d}{dN}\ln|f_{RR}| + \frac{1-\mathcal{H}'/\mathcal{H}^2}{2-\mathcal{H}''/\mathcal{H}^3}.$$

$$\chi \equiv \sqrt{\frac{a^2}{3f_{RR}}\frac{1-\mathcal{H}'/\mathcal{H}^2}{2-\mathcal{H}''/\mathcal{H}^3}}. \qquad \frac{1-\mathcal{H}'/\mathcal{H}^2}{2-\mathcal{H}''/\mathcal{H}^3} \sim 1$$

$f_{RR} > 0$ → Oscillation, $f_{RR} < 0$ → Instability.

$$f_{\rm eff} \simeq -\frac{1}{2} + \frac{9}{2}\left(2 - \frac{\mathcal{H}''}{\mathcal{H}^3}\right)\frac{\mathcal{H}^2 f_{RRR}}{a^2 f_{RR}}.$$

Here, $\mathcal{H}''/\mathcal{H}^3 \simeq 1/2$ is held in the matter dominant era.

$f_{RR} > 0$

$f_{RRR} > 0$ → Growing oscillation, $f_{RRR} < 0$ → Decaying oscillation.

Summary

- Cosmological perturbations in the ΛCDM model, *k*-essence model and F(R) gravity model have been considered.
- The matter density perturbation depends on the scale at sub-leading order in each models.
- In addition to the quasi-static solutions, *k*-essence model has the oscillating solutions.
- Viable F(R) gravity models cannot be distinguished from the ΛCDM model by evaluating the growth rate of the structure formation when we fit their background evolution to the observational results.
- A sufficient conditions for the fast fluctuating mode to be the decaying oscillating solutions are $f_{RR} > 0$ and $f_{RRR} < 0$.

Perturbative equations in k-essence model

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \quad \phi \to \phi + \delta\phi.$$

Einstein equations

(0,0)

$$-6H^2\Phi - 2\frac{k^2}{a^2}\Psi - 6H\dot{\Psi} = \kappa^2\Big\{(-K_{,X} + \dot{\phi}^2 K_{,XX})(\dot{\phi}^2\Phi + \dot{\phi}\delta\dot{\phi}) - (K_{,\phi} - \dot{\phi}^2 K_{,X\phi} - 2\dot{\phi}K_{,X}\partial_0)\delta\phi + 2\Phi\dot{\phi}^2 K_{,X} - \delta\rho\Big\},$$

(0,i)

$$2\partial_i(H\Phi + \dot{\Psi}) = \kappa^2\left\{\dot{\phi}K_{,X}\partial_i\delta\phi + (\rho + p)\delta u_i\right\},$$

(i,j)

$$a^{-2}\partial_i\partial_j(\Phi - \Psi) = 0, \quad (i \neq j),$$

(i,i)

$$\left(\frac{k^2}{a^2} + \frac{\partial_i\partial_i}{a^2} - 2H\partial_0 - 4\dot{H} - 6H^2\right)\Phi - \left(\frac{k^2}{a^2} + \frac{\partial_i\partial_i}{a^2} + 2\partial_0\partial_0 + 6H\partial_0\right)\Psi = \kappa^2\left\{-K_{,X}(\dot{\phi}^2\Phi + \dot{\phi}\delta\dot{\phi}) - K_{,\phi}\delta\phi + \delta p\right\},$$

Equation of motion of the scalar field

$$\begin{aligned}(\dot{\phi}^2 K_{,XX} + K_{,X})\delta\ddot{\phi} = &-3\dot{\phi}K_{,X}\dot{\Psi} + (-\dot{\phi}K_{,X}\partial_0 - \dot{\phi}^3 K_{,XX}\partial_0 - 6H\dot{\phi}K_{,X} - 2\ddot{\phi}K_{,X} \\ &- 2\dot{\phi}\dot{K}_{,X} - 3H\dot{\phi}^3 K_{,XX} - 3\dot{\phi}^2\ddot{\phi}K_{,XX} - \dot{\phi}^3\dot{K}_{,XX} + \dot{\phi}^2 K_{,\phi X})\Phi \\ &- (3HK_{,X} + \dot{K}_{,X} + 3H\dot{\phi}^2 K_{,XX} + 2\dot{\phi}\ddot{\phi}K_{,XX} + \dot{\phi}^2\dot{K}_{XX})\delta\dot{\phi} \\ &+ \left(-K_{,X}\frac{k^2}{a^2} - 3H\dot{\phi}K_{,X\phi} - \dot{\phi}\dot{K}_{,X\phi} - \ddot{\phi}K_{,X\phi} + K_{,\phi\phi}\right)\delta\phi.\end{aligned}$$

Equations of continuity

(0)

$$\delta\dot{\rho} + 3H(\delta\rho + \delta p) + a^{-2}\partial_i\{(\rho + p)\delta u_i\} + 3\dot{\Psi}(\rho + p) = 0,$$

(i)

$$a^{-3}\partial_0\{a^3(\rho + p)\delta u_i\} + \partial_i\delta p - (\rho + p)\partial_i\Phi = 0.$$

$\delta = \delta\rho/\rho$

$c_s^2 \equiv \delta p/\delta\rho$

Perturbative equations

Newtonian gauge $ds^2 = a^2(\eta)[(1+2\Phi)d\eta^2 - (1+2\Psi)\sum_{i=1}^{3} dx^i dx^i]$

Einstein equations

(0,0)
$$(1+f_R)\{-k^2(\Phi+\Psi) - 3\mathcal{H}(\Phi'+\Psi') + (3\mathcal{H}'-6\mathcal{H}^2)\Phi - 3\mathcal{H}'\Psi\}$$
$$+ f_R'(-9\mathcal{H}\Phi + 3\mathcal{H}\Psi - 3\Psi') = \kappa^2\rho a^2\delta$$

(i,i)
$$(1+f_R)\{\Phi''+\Psi'' + 3\mathcal{H}(\Phi'+\Psi') + 3\mathcal{H}'\Phi + (\mathcal{H}'+2\mathcal{H}^2)\Psi\}$$
$$+f_R'(3\mathcal{H}\Phi - \mathcal{H}\Psi + 3\Phi') + f_R''(3\Phi - \Psi) = c_s^2\kappa^2\rho a^2\delta$$

(0,i)
$$(1+f_R)\{\Phi'+\Psi'+\mathcal{H}(\Phi+\Psi)\} + f_R'(2\Phi-\Psi) = -\kappa^2\rho a^2(1+w)v$$

(i,j)
$$\Phi - \Psi - \frac{2f_{RR}}{a^2(1+f_R)}\{3\Psi'' + 6(\mathcal{H}'+\mathcal{H}^2)\Phi + 3\mathcal{H}(\Phi'+3\Psi') - k^2(\Phi-2\Psi)\} = 0$$

The equations of continuity

(0)
$$3\Psi'(1+w) - \delta' + 3\mathcal{H}(w-c_s^2)\delta + k^2(1+w)v = 0,$$

(i)
$$\Phi + \frac{c_s^2}{1+w}\delta + v' + \mathcal{H}v(1-3w) = 0.$$

$\delta = \delta\rho/\rho$

$c_s^2 \equiv \delta p/\delta\rho$