Restricting profile function of hedgehog Skyrmion

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Abstract

The profile function for the hedgehog Skyrmion is investigated. After discussing how the form of the profile function is restricted by the field equation, the static energy is numerically calculated. It is found that the profile function considered here sometimes give the static energy smaller than the previous ones.

- **Skyrme model** (T. H. R. Skyrme, Nucl.Phys.**31**,1962)

Lagrangian Density:

$$\mathcal{L}_S = \frac{F_\pi^2}{16}\mathrm{tr}\left(\partial_\mu U)(\partial^\mu U^\dagger\right) + \frac{1}{32e^2}\mathrm{tr}\left([\partial_\mu U U^\dagger\,,\,\partial_\nu U U^\dagger][\partial^\mu U U^\dagger\,,\,\partial^\nu U U^\dagger]\right),$$
$$U = U(x) \in SU(2), \quad (F_\pi, e\text{: const.})$$

With defining $R_\mu = (\partial_\mu U)\,U^\dagger$,

Field equation:

$$\partial_\mu\left(R^\mu + \frac{1}{4}\,[R^\nu,\ \ [R_\nu,\ R^\mu]]\right) = 0.$$

Energy E:

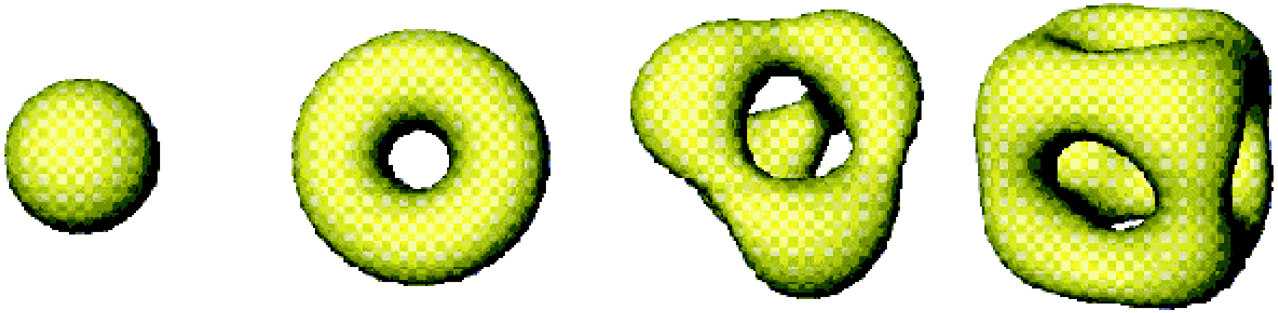
$$E = \frac{1}{12\pi}\int\left[-\frac{1}{2}\mathrm{tr}\,(R_i R_i) - \frac{1}{16}\mathrm{tr}\left(\left[R_i,\ R_j\right]\left[R_i,\ R_j\right]\right)\right] d^3x$$

Baryon number B:

$$B = \frac{\varepsilon_{ijk}}{24\pi^2} \int \mathrm{tr}\left(R_i R_j R_k\right) d^3x.$$

Energy should satisfy

$$E \geqq |B|$$



$B=1$ $B=2$ $B=3$ $B=4$.

Baryon density isosurfaces for each B (Polyhedral).

(R.A.Battye, P.M.Sutcliffe Phys. Rev. Lett.**81**, 4798 (1998))

• Spherically Symmetric Ansatz

Spherically symmetric ansatz:

$U = \exp\left[if(r)\,(\boldsymbol{x}\cdot\boldsymbol{\tau})\right], \quad r = |\boldsymbol{x}|, \quad \boldsymbol{\tau}$: Pauli matrices,

Field equation:

$$\left[r^2 + 2\sin^2 f(r)\right]\frac{d^2 f(r)}{dr^2} + 2r\left(\frac{df(r)}{dr}\right) + \sin 2f(r)\left[\left(\frac{df(r)}{dr}\right)^2 - 1 - \frac{\sin^4 f(r)}{r^2}\right] = 0$$

Introducing $z = \dfrac{r^2}{r^2+2}, \quad v(z) \equiv \tan^2 f(z),$

Field equation(Algebraic):

$$\frac{d^2 v}{dz^2} - \frac{1}{2}\left[\frac{3}{v-1} + \frac{1}{v} - \frac{1}{v-z}\right]\left(\frac{dv}{dz}\right)^2 + \frac{1}{2}\left[\frac{1}{z-1} + \frac{1}{z} - \frac{2}{v-z}\right]\frac{dv}{dz} + \frac{v\left[v(z+1) - 2z\right]}{2z^2(z-1)^2(z-v)} = 0$$

Energy:

$$E = \frac{1}{3\sqrt{2}\pi}\int_0^1 \left[\frac{(z-v)\sqrt{z(1-z)}}{v(v-1)^3}\left(\frac{dv}{dz}\right)^2 + \frac{v(3zv - 4z + v)}{2(v-1)^2\sqrt{z^3(1-z)^3}}\right] dz$$

• Solution of the field equation

Skyrme field eq. under the spherically symmetric ansatz:

$$\frac{d^2v}{dz^2}-\frac{1}{2}\left[\frac{3}{v-1}+\frac{1}{v}-\frac{1}{v-z}\right]\left(\frac{dv}{dz}\right)^2+\frac{1}{2}\left[\frac{1}{z-1}+\frac{1}{z}-\frac{2}{v-z}\right]\frac{dv}{dz}+\frac{v\left[v(z+1)-2z\right]}{2z^2\left(z-1\right)^2\left(z-v\right)}=0$$

$$2v\left(v-1\right)\left(z-v\right)\frac{d^2v}{dz^2}+\left(3v^2-4zv+z\right)\left(\frac{dv}{dz}\right)^2+\frac{v\left(v-1\right)}{z\left(z-1\right)}\left(4z^2-2zv-3z+v\right)\frac{dv}{dz}+\frac{v^2\left(v-1\right)}{z^2\left(z-1\right)^2}\left[v\left(z+1\right)-2z\right]=0$$

If we assume $v(z)$ of the form

$$v(z)=\sum_{j=0}^{\infty}v_j\left(z-z_0\right)^{j-\alpha},\qquad(\alpha>0),$$

we obtain

$$-2\alpha\left(\alpha+1\right)v_0^4\left(z-z_0\right)^{-(4\alpha+2)}+3\alpha^2v_0^4\left(z-z_0\right)^{-(4\alpha+2)}+\mathcal{O}\left(\left(z-z_0\right)^{-(4\alpha+2)+1}\right)=0 \qquad \therefore\ \alpha=2.$$

Substituting $v = \sum_{j=0}^{\infty} v_j\,(z - z_0)^{j-2}$ into the equation v_j's($j = 1, 2, 3, ...$) are determined except v_0.

$$v_0 = \text{arbitrary}, \quad v_1 = \frac{-v_0 + 2v_0 z_0}{2z_0(z_0 - 1)}, \quad v_2 = \frac{v_0 + 4v_0 z_0^2 - 16z_0^2(z_0 - 3)(z_0 - 1)^2}{48z_0^2(z_0 - 1)^2},$$

$$v_3 = \frac{-32z_0^3(z_0 - 1)^3 + v_0(1 - 2z_0 - 4z_0^2)}{96z_0^3(z_0^3 - 1)^3},$$

$$v_4 = \frac{-256z_0^5(z_0 - 1)^4(2z_0 - 3) + 32v_0 z_0^2(z_0 - 1)^2(2z_0^3 - z_0^2 + 3) + v_0^2(53 - 80z_0 + 96z_0^2 + 160z_0^3 - 16z_0^4)}{3840v_0 z_0^4(z_0^4 - 1)^4}.$$

$$v_5 = ...$$

All v_js are determined.

At spatial origin ($z_0 = 0$):

$$v(z) = \sum_{j=0}^{\infty} v_j z^{j+\beta}.$$

Candidates for α are

$$\beta < 0, \quad \beta = 0, \quad 0 < \beta < 1, \quad \beta = 1, \quad \beta > 1.$$

Consistent case: $\beta = 1$.

Ex.($\alpha < 0$ case)

$$2v(v-1)(z-v)\frac{d^2v}{dz^2} + \left(3v^2 - 4zv + z\right)\left(\frac{dv}{dz}\right)^2 + \frac{v(v-1)}{z(z-1)}\left(4z^2 - 2zv - 3z + v\right)\frac{dv}{dz} + \frac{v^2(v-1)}{z^2(z-1)^2}\left[v(z+1) - 2z\right] = 0$$

Leading terms $\Downarrow$ $v(z) = v_0 z^{-\alpha}$

$$-2v_0^4 z^{4\beta}\beta(\beta-1) + 3v_0^4 z_{4\beta}\beta^2 - \beta v_0^4 z^{4\beta} + v_0^4 z^{4\beta} = 0$$

$$v_0^4 z^{4\beta}\left(\beta^2 + \beta + 1\right) = 0$$

$\therefore$ No solution satisfying $\beta < 0$.

At spatial infinity ($z_0 = 1$):

$$v(z) = \sum_{j=0}^{\infty} v_j \, (z-1)^{j-\gamma}$$

Leading order $\rightarrow$ $\gamma = 2$.

$z = 0, 1$ correspond $r = 0, \infty$ $\quad \because z = \dfrac{r^2}{r^2+2}$

Boundary condition: $\quad f(0) = \pi, \quad f(\infty) = 0 \Rightarrow$ Baryon number $B = 1$

The solution;

$$v(z) = \frac{z(1-z)^2}{(z-z_0)^2} \sum_{j=0}^{\infty} w_j \, (z-z_0)^j$$

satisfies the boundary condition.

Substituting $v(z) = \dfrac{z\,(z-1)^2}{(z-z_0)^2} \sum_{j=0}^{\infty} w_j\,(z-z_0)^j$ into the field equation

$$\frac{d^2v}{dz^2} - \frac{1}{2}\left[\frac{3}{v-1} + \frac{1}{v} - \frac{1}{v-z}\right]\left(\frac{dv}{dz}\right)^2 + \frac{1}{2}\left[\frac{1}{z-1} + \frac{1}{z} - \frac{2}{v-z}\right]\frac{dv}{dz} + \frac{v\,[v(z+1)-2z]}{2z^2\,(z-1)^2\,(z-v)} = 0$$

w_j's$(j = 0, 1, 2 ...)$ are determined except w_0.

$$w_0 = \text{arbitrary}, \quad w_1 = \frac{v_0(4z_0-1)}{2z_0(1-z_0)}, \quad w_2 = \frac{(148z_0^2 - 72z_0 + 25)w_0 - 16z_0(z_0-3)}{48z_0^2(1-z_0)^2},$$

$$w_3 = \frac{408z_0^3 - 292z_0^2 + 200z_0 - 51)w_0 - 32z_0(2z_0^2 - 9z_0 + 3)}{96z_0^3(z_0^3-1)^3},$$

$$w_4 = \frac{1}{3840w_0z_0^4(z_0^4-1)^4}\Big[(21104z_0^4 - 19840z_0^3 + 20176z_0^2 - 10200z_0 + 2093)w_0^2 - (3776z_0^4 - 23008z_0^3 + 15260z_0^2 + 3936z_0)w_0 + 768z_0^3 - 512z_0^4\Big].$$

We seek the solution minimizing the energy:

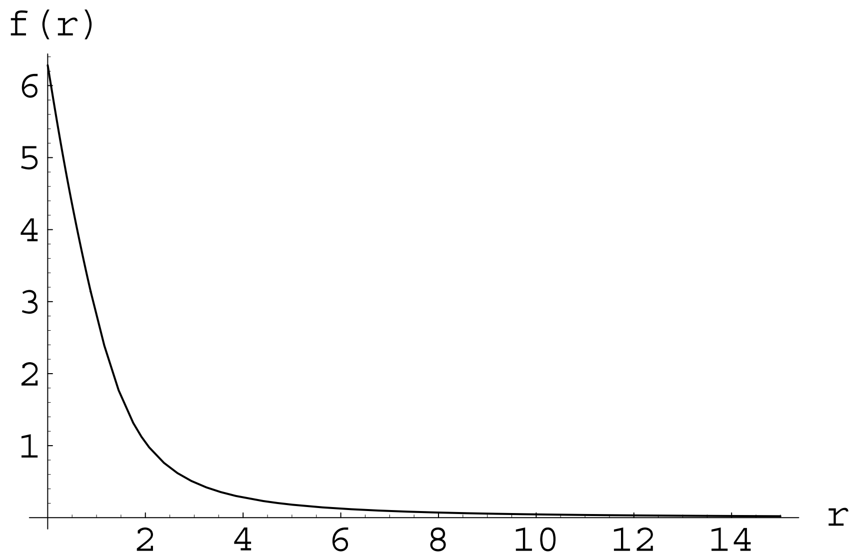
$$v(z) = \frac{z(1-z)^2}{(z-z_0)^2} w(z)$$

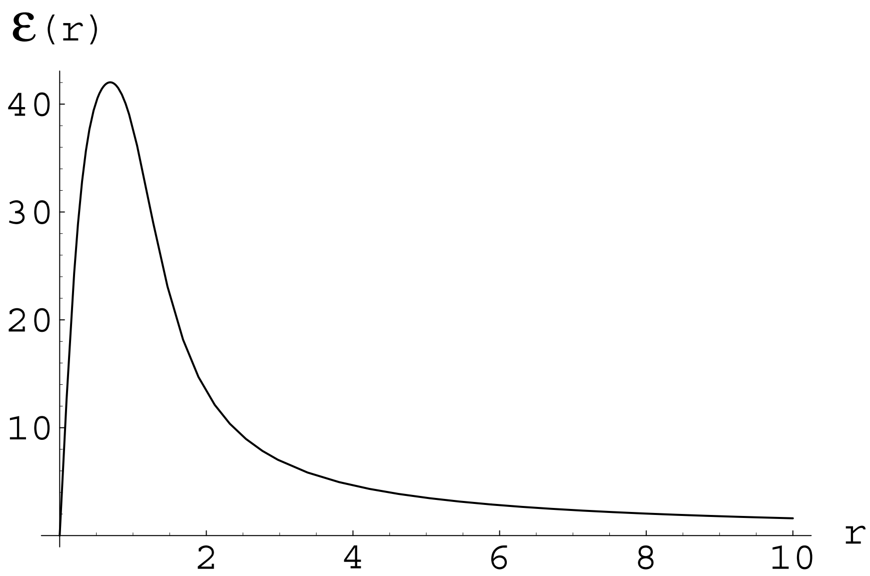
Since we defined $v(z) \equiv \tan^2 f(r)$ we can choose profile function $f(r)$ as

$$f(r) = \pi - \text{Arctan}\sqrt{v(z)}, \quad (0 \le z \le z_0),$$
$$f(r) = \text{Arctan}\sqrt{v(z)}, \quad (z_0 \le z \le 1).$$

In the simplest case $w(z) = w_0$(const.), we find

$$E = 1.23186 \quad \text{at} \quad w_0 = 0.673, \quad z_0 = 0.279$$





• Summary and discussion

- We obtained a solution which is of Laurent series type of the Skyrme model under the spherically symmetric ansatz.

$$\frac{d^2v}{dz^2} - \frac{1}{2}\left[\frac{3}{v-1} + \frac{1}{v} - \frac{1}{v-z}\right]\left(\frac{dv}{dz}\right)^2 + \frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{z} - \frac{2}{v-z}\right]\frac{dv}{dz} + \frac{v\left[v(z+1)-2z\right]}{2z^2(z-1)^2(z-v)} = 0$$

$$U(x) = \exp\left[if(r)\,(\boldsymbol{x}\cdot\boldsymbol{\tau})\right], \quad z = \frac{r^2}{r^2+2},$$

$$v(z) = \tan^2 f(r), \\ = \frac{z(1-z)^2}{(z-z_0)^2}\sum_{j=0}^{\infty} w_j (z-z_0)^j .$$

- The solution satisfies the boundary condition

$$f(0) = \pi, \quad f(\infty) = 0 \Rightarrow \text{Baryon number } B = 1$$

Energy:

$$E = \frac{1}{3\sqrt{2}\pi}\int_0^1 \left[\frac{(z-v)\sqrt{z(1-z)}}{v(v-1)^3}\left(\frac{dv}{dz}\right)^2 + \frac{v(3zv-4z+v)}{2(v-1)^2\sqrt{z^3(1-z)^3}}\right] dz$$

$$v(z) = \frac{z\,(1-z)^2}{(z-z_0)^2} \underbrace{\sum_{j=0}^{N} w_j\,(z-z_0)^j}_{\equiv\; w(z)}$$

- Local minimum of Energy E

$N = 0:\quad E = 1.23186$ at $w_0 = 0.673, \quad z_0 = 0.279.$

$N = 1:\quad E = 1.23215$ at $w_0 = 0.670, \quad z_0 = 0.275.$

$N = 2:\quad E = 1.34000$ at $w_0 = 0.485, \quad z_0 = 0.327.$

$N = 3:\quad E = 1.34234$ at $w_0 = 0.249, \quad z_0 = 0.249.$

$E = 1.23186$

This E is smaller than that obtained by Battye-Sutcliffe ($E = 1.2322$)

R. A. Battye and P. M. Sutcliffe, *Phys. Rev. Lett.* **79**, (1997) 363.

- Comparison with numerical analysis

Adkins, Nappi and Witten solved the field equation numerically and

calculated $\left\{ \begin{array}{l} \text{the isoscalar electric mean square radius} \sqrt{\langle r^2 \rangle_{I=0}} \\ \text{the isoscalar magnetic mean square radius} \sqrt{\langle r^2 \rangle_{M,I=0}} \end{array} \right.$

defined by

$$\sqrt{\langle r^2 \rangle_{I=0}} = \frac{2}{eF_\pi} \sqrt{-\frac{2}{\pi} \int_0^\infty dr\ r^2 \sin^2 f(r) f'(r)},$$

$$\sqrt{\langle r^2 \rangle_{M,I=0}} = \frac{2}{eF_\pi} \sqrt{\frac{\int_0^\infty dr\ r^4 \sin^2 f(r) f'(r)}{\int_0^\infty dr\ r^2 \sin^2 f(r) f'(r)}}.$$

G. S. Adkins, C. R. Nappi, and E. Witten, (*Nucl. Phys.* **B 228**, (1983) 552)

Results of Adkins, Nappi and Witten:

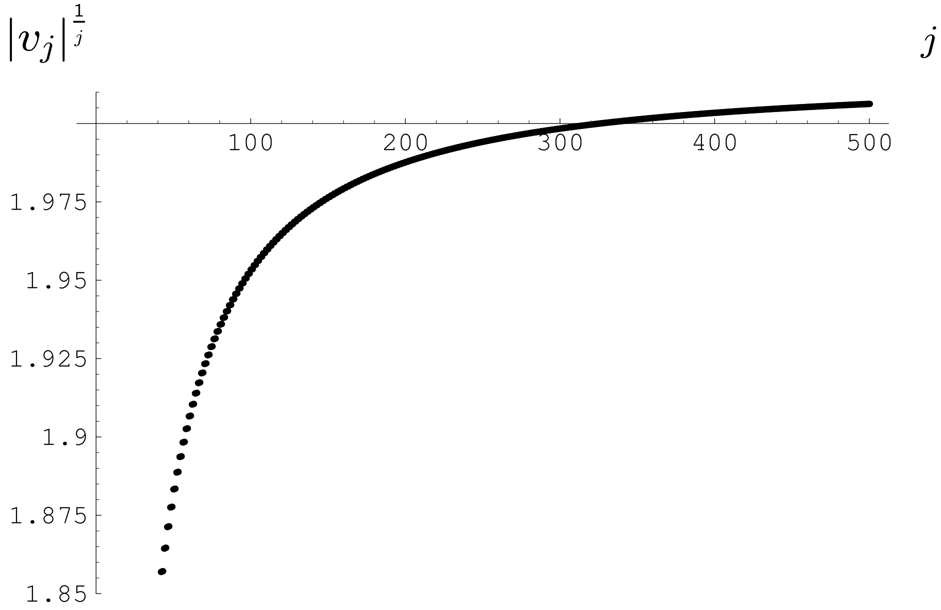
$$M = 36.5\frac{F_\pi}{e},$$
$$e = 5.45, \qquad F_\pi = 129 \text{ MeV},$$
$$\sqrt{\langle r^2\rangle_{I=0}} = 0.59 \text{ fm}, \quad \sqrt{\langle r^2\rangle_{M,I=0}} = 0.92 \text{ fm}.$$

Our result:

$$M = 36.5\frac{F_\pi}{e},$$
$$e = 5.48, \qquad F_\pi = 130 \text{ MeV},$$
$$\sqrt{\langle r^2\rangle_{I=0}} = 0.586 \text{ fm}, \quad \sqrt{\langle r^2\rangle_{M,I=0}} = 0.920 \text{ fm}.$$

Here, input data are $M_{\mathsf{N}} = 939\text{MeV}$, delta $M_{\Delta} = 1232\text{MeV}$.

$z_0 = 1/2$ for example, up to v_{500}, all v_js are consistent:



Radius of convergence of $v(z)$:

$$R_c = |v_{500}|^{-\frac{1}{500}} = \frac{1}{2.00626}$$

$$\Downarrow$$

$v(z)$ is regular in this region

$$z = \frac{r^2}{r^2 + 2}$$

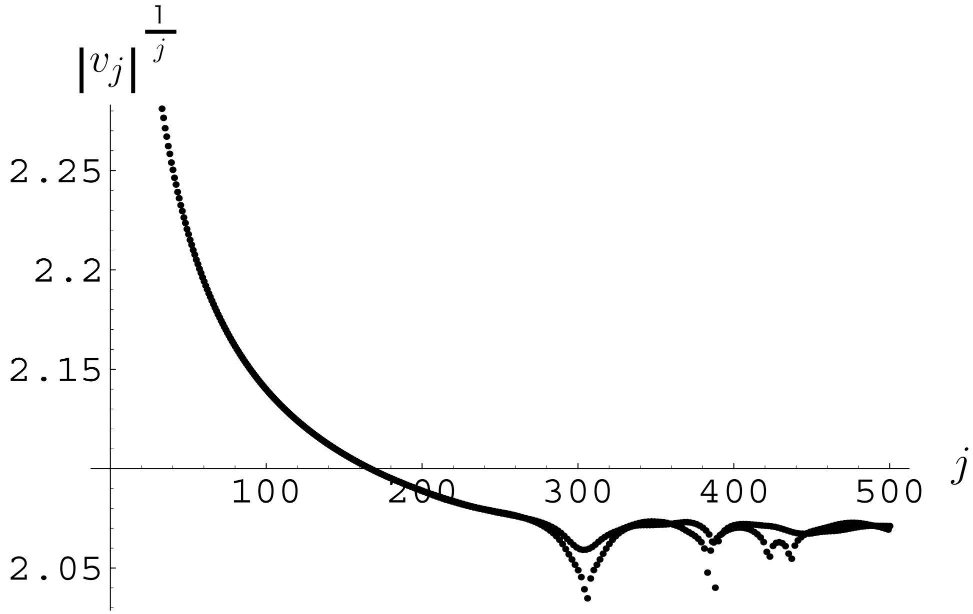
$$(0 \leq r \leq \infty) \Rightarrow (0 \leq z \leq 1)$$

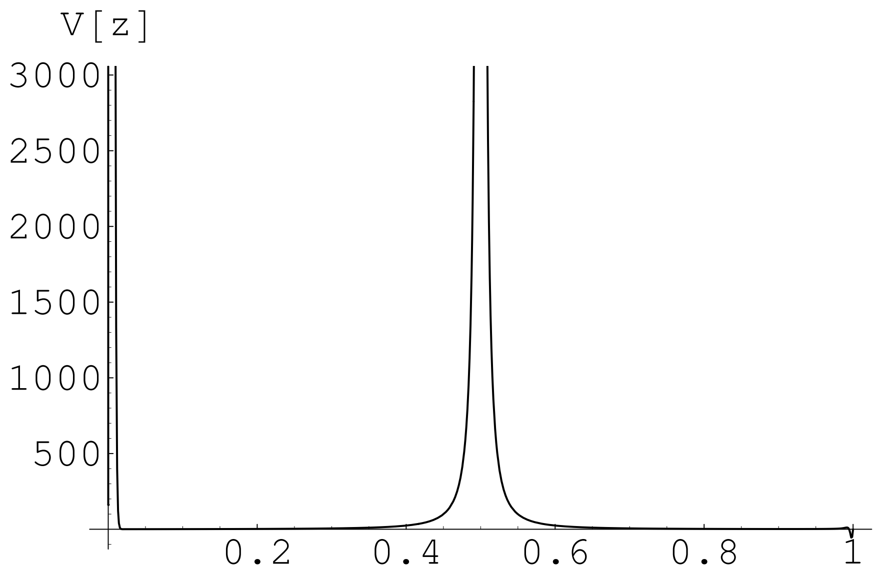
For example, $v_1, v_2, v_3, \ldots$ with $v_0 = -2$, and $z_0 = \frac{1}{2}$ are

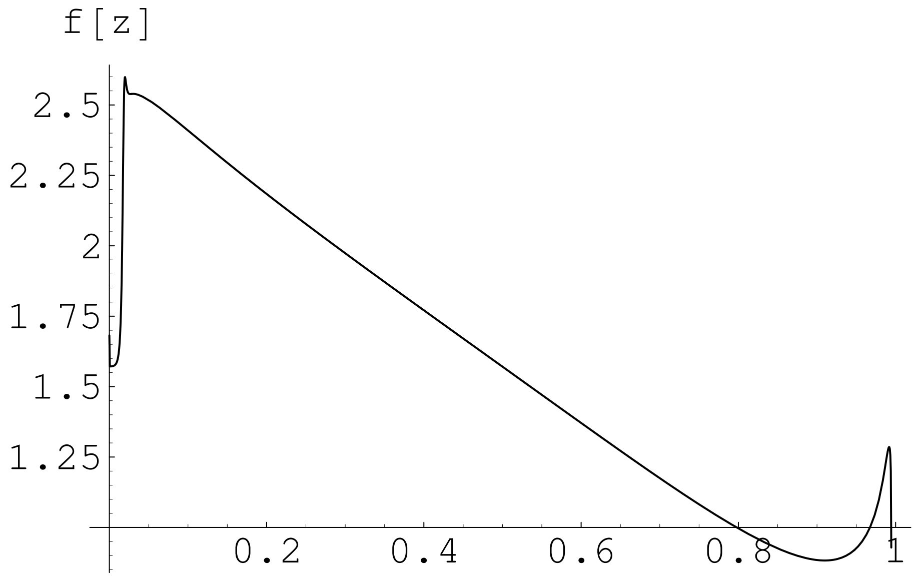
$$V(z) = \frac{z\,(z-1)^2}{(z-z_0)^2} \sum_{j=0}^{500} v_j \left(z - \frac{1}{2}\right)^j$$

Radius of convergence:

$$R_c = \left| v_{500} \right|^{-\frac{1}{500}} = \frac{1}{2.07256}$$







Inputs of nucleon mass M_{N} and delta M_{Δ} are given by M_{N}=939MeV and M_{Δ}=1232MeV. M_{N} and delta M_{Δ} are given by

$$M_{\mathrm{N}} = M + 3/(8\lambda), \quad M_{\Delta} = M + 15/(8\lambda).$$

$$M = \frac{3\pi^2 F_\pi}{e} E,$$

$$E = \frac{1}{3\pi} \int_0^\infty dr\ r^2 f'(r)^2 + 2\sin^2 f(r)[f'(r)^2 + 1] + \frac{\sin^4 f(r)}{r^2},$$

$$\lambda = \frac{2\pi}{3e^3 F_\pi} \Lambda,$$

$$\Lambda = 8 \int_0^\infty dr\ r^2 \sin^2 f(r) \left[1 + \left(\frac{df(r)}{dr}\right)^2 + \frac{\sin^2 f(r)}{r^2}\right],$$