I. Introduction

II. Wilsonian RG and Power Counting

III. Wilsonian RG and Pionless NEFT

IV. Summary

This talk is based on works in collaboration with Koji Harada (Kyushu University)
Introduction

Characteristic Features of Nuclear Interaction

- Very strong

- Depends on the states of nucleons (spin, isospin)

- Binding energy/nucleon $\sim 8\text{MeV}$ (stable nuclei)

Two Nucleon System

- Very shallow ($E_b \sim 2.2\text{MeV}$) bound state; deuteron in $^3S_1$ channel

- Very large ($1/a(^1S_0) \sim -8.3\text{MeV}$) scattering length in $^1S_0$ channel
Fine-Tuning

Range of NN potential, $R \sim \frac{1}{m_\pi}$, uncertainty principle $p \gtrsim \frac{1}{R} \sim m_\pi$

$$H = T + V, \quad T \sim \frac{m_\pi^2}{M} \sim \frac{(140)^2}{1000} \sim 20\text{MeV}, \quad M : \text{nucleon mass}$$

$H \sim T$ (assuming that there are no delicate cancellations)

$2.2\text{MeV} \ll 20\text{MeV}$ (energy scale)

$8\text{MeV} \ll 140\text{MeV}$ (momentum scale) Finely-Tuned System!

Potential Model Approach

Few nucleon systems — “Realistic ” NN Potential, $V_{NN}$

CD Bonne, AV$_{18}$, Paris, Nijimegen I,II, · · · excellent agreement with experiments
Introduction

Problems in Potential Model Approach

- Systematic improvement (estimation of errors) seems difficult
- Why are 3-body, 4-body,.. forces weaker than 2-body force?
- Relation to QCD e.g. chiral symmetry, quark mass dependence of nuclear force
- Off-shell ambiguity (could be a problem in multi-nucleon systems)

We need a model-independent description!
Effective Field Theory (EFT)

**Weinberg’s “folk” theorem**

The most general Lagrangian consistent with assumed symmetry

\[ \downarrow \]

The most general S-matrix consistent with general principles of QFT
(analyticity, perturbative unitarity, cluster decomposition and the assumed symmetry principles)
Nontrivial Assumptions

- Identification of relevant low energy degrees of freedom
- **Power Counting**: Rule to organize the contributions

Usually,

**Naive Dimensional Analysis (NDA)** \(\sim\) Naturalness

\[ \rightarrow \text{Classical Dimensional Analysis} \]

Contribution from an operator of mass dimension \(d_i\): 
\[ a_i \left( \frac{Q}{\Lambda_0} \right)^{d_i}, \quad a_i \sim O(1) \]

\[ \Rightarrow d_i \rightarrow \text{larger, more suppressed} \]

In the case in which quantum corrections are small, NDA should work
Introduction

However,

Nuclear system is **Nonperturbative & Finely-tuned**

\[ \Downarrow \]

NDA would **not** work

How do we find the correct Power Counting?
Wilsonian RG and Power Counting

What is Power Counting? → Dimensional Analysis (Quantum theoretical)

**Wilsonian RG** allows us to determine the dimension of an operator

Legendre Flow Equation:

\[
\frac{d\Gamma_\Lambda}{dt} = \frac{i}{2} \text{Tr} \tilde{\partial}_t \left[ \ln \left( (\Gamma_2) + R_\Lambda \right) \right], \quad (\Gamma_2)_{nm} \equiv \frac{\delta^2 \Gamma_\Lambda}{\delta \Phi_n \delta \Phi_m}, \quad \Lambda = \Lambda_0 e^{-t}
\]

(Effective average action, \( \Gamma_\Lambda \): Usual effective action + IR cutoff.)

RG transformation generates all operators allowed by the symmetry

\[
\Gamma_\Lambda = \int d^4x \sum_{i=\text{all}} G_i(\Lambda) \mathcal{O}_i, \quad G_i(\Lambda) \equiv \frac{g_i(\Lambda)}{\Lambda^{d_i-4}}, \quad d_i : G_i \text{ has dimension } (4 - d_i).
\]
Wilsonian RG and Power Counting

One-Loop Structure

$P^{-1}$: full propagator (with IR cutoff)

$$\frac{d\Gamma\Lambda}{dt} = \frac{i}{2} \text{Tr} \tilde{\partial}_t \left[ \ln P + (P^{-1} \mathcal{F}) - \frac{1}{2} (P^{-1} \mathcal{F})^2 + \frac{1}{3} (P^{-1} \mathcal{F})^3 + \ldots \right]$$

Graphically

$$\Rightarrow \beta_i(g) = -\Lambda \frac{dg_i}{d\Lambda}, \quad \text{(An infinite \# of coupled first-order differential equations)}$$
Wilsonian RG and Power Counting

Fixed Point and Scaling Dimension

- **Fixed point:** \( \beta_i(g^*) = 0 \)

Linearized RG equation:

\[
\delta g_i = g_i^* + \delta g_i, \quad -\Lambda \frac{d}{d\Lambda} \delta g_i = \left. \frac{d\beta_i(g)}{dg_j} \right|_{g=g^*} \delta g_j
\]

Diagonalize \( A_{ij}(g^*) \)

Solution near the fixed point

\[
\frac{du_i}{dt} = \nu_i u_i, \quad u_i(\Lambda) = u_i(\Lambda_0) \left( \frac{\Lambda_0}{\Lambda} \right)^{\nu_i} \quad \Lambda_0: \text{physical cutoff of EFT}
\]

\( \nu_i: \text{Scaling Dimension} \) (of the coupling)

\( \nu > 0 \Rightarrow \text{Relevant}, \quad \nu = 0 \Rightarrow \text{Marginal}, \quad \nu < 0 \Rightarrow \text{Irrelevant} \)
Wilsonian RG and Power Counting

**Dimensional Analysis** ($\Lambda_0$-dependence)

Dimensionful coupling:

$$G_i(\Lambda) = \frac{g_i(\Lambda)}{\Lambda^{d_i-D}} \sim \frac{g_i^*}{\Lambda^{d_i-D}} + \sum_k c_{ik} \frac{\Lambda_0^{\nu_k}}{\Lambda^{d_i-D+\nu_k}} \text{ small constant}$$

- Right on a fixed point ($c_{ik} = 0$): ⇒ **Scale Invariant theory**
  (fixed point value, $G_i^*$ itself does not contribute to Dimensional Analysis)

- Vicinity of a fixed point ($c_{ik} \neq 0$): ⇒ $k$-th term in the sum acts as a coupling constant with dimension $\nu_k$

**Scaling dimensions are not prescribed, but determined by the theory itself.**
Wilsonian RG and Pionless NEFT

**Two-Nucleon Sector**

Due to the NR feature: No anti-nucleons (as explicit degrees of freedom)

$→$ Nucleon number is conserved.

$→ (N^\dagger N)^3, (N^\dagger N)^4, \cdots$ do not contribute to two-nucleon sector.

$$\left. \frac{d\Gamma_\Lambda}{dt} \right|_{4N} = -\frac{i}{4} \text{Tr} \tilde{\partial}_t \left[(\mathcal{P}^{-1}\mathcal{F})^2 \right]_{4N}$$

**Exact equation**

\[
\begin{align*}
\frac{d}{d\Lambda} & \quad = \quad \frac{dR}{d\Lambda} \\
\bullet & \quad = \text{four-nucleon vertices} \\
\otimes & \quad = \frac{dR}{d\Lambda}
\end{align*}
\]
Wilsonian RG and Pionless NEFT

Truncation

Some kind of approximation is necessary for practical calculations.

⇒ Truncate the operator space

- **Expectation**: Importance of operators may be different from the one given by NDA, their orderings would not change. The lower an operator’s dimension, the less important it is.

- **Reliability** of the truncation may be checked by the stability against the enlargement of the operator space.
**Pionless NEFT**

**Wilsonian RG and Pionless NEFT**

**Truncation:** by number of derivatives, $O(\nabla^2, \partial_t)$ (regarding $\nabla^2 \sim \partial_t$)

$$\Gamma_\Lambda = \int d^4x N^\dagger \left( i\partial_t + \frac{\nabla^2}{2M} \right) N - C_0^{(S)} O_0^{(S)} + C_2^{(S)} O_2^{(S)} + 2B^{(S)} O_2^{(SB)}$$

$$O_0^{(S)} = \left( NT P_a^\dagger \left( 1S_0 \right) N \right)^\dagger \left( NT P_a^\dagger \left( 1S_0 \right) N \right), \quad O_2^{(S)} = \left[ \left( NT P_a^\dagger \left( 1S_0 \right) N \right)^\dagger \left( NT P_a^\dagger \left( 1S_0 \right) \nabla^2 N \right) + h.c. \right],$$

$$O_2^{(SB)} = \left\{ \left( NT P_a^\dagger \left( 1S_0 \right) \left( i\partial_t + \frac{\nabla^2}{2M} \right) N \right)^\dagger \left( NT P_a^\dagger \left( 1S_0 \right) N \right) + h.c. \right\} _{\text{Redundant operator}}$$

"Redundant Operators": operators which can be eliminated by field redefinition, or use of EOM

**These must be included in Wilsonian RG analysis**
Wilsonian RG and Pionless NEFT

**RG Flow (in $^1S_0$-channel)**

Dimensionless couplings: $x \equiv \frac{M \Lambda}{2\pi^2} C_0^{(S)}$, $y \equiv \frac{M \Lambda^3}{2\pi^2} 4 C_2^{(S)}$, $z \equiv \frac{\Lambda^3}{2\pi^2} B^{(S)}$

\[
\frac{dx}{dt} = -x - \left[ x^2 + 2xy + y^2 + 2xz + 2yz + z^2 \right]
\]

\[
\frac{dy}{dt} = -3y - \left[ \frac{1}{2} x^2 + 2xy + \frac{3}{2} y^2 + yz - \frac{1}{2} z^2 \right]
\]

\[
\frac{dz}{dt} = -3z + \left[ \frac{1}{2} x^2 + xy + \frac{1}{2} y^2 - xz - yz - \frac{3}{2} z^2 \right]
\]
Wilsonian RG and Pionless NEFT

Fixed Points

\(1S_0\) channel:
\[ (x^*, y^*, z^*) = (0, 0, 0), \quad (-1, -\frac{1}{2}, \frac{1}{2}), \quad (-9, \frac{15}{2}, -\frac{3}{2}) \]

\(3S_1-3D_1\) channel:
\[ (x'^*, y'^*, z'^*, w'^*) = (0, 0, 0, 0), \quad (-1, -\frac{1}{2}, \frac{1}{2}, 0), \quad (-9, \frac{15}{2}, -\frac{3}{2}, 0) \]
Figure 1: The RG flow for the $^1S_0$ channel projected on to the $C_0^{(S)} – C_2^{(S)}$ plane.

Figure 2: The RG flow for the $^3S_1–^3D_1$ channel projected on to the $C_0^{(T)} – C_2^{(T)}$ plane.
Wilsonian RG and Pionless NEFT

Scaling Dimension

Trivial fixed point ($^1S_0$ channel)

\[
\frac{d}{dt} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}
\]

\[
\nu_1 = -1 : u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \nu_2 = -3 : u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \nu_3 = -3 : u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

Agree with NDA

All \textbf{Irrelevant} $\Rightarrow$ suppression $\Rightarrow$ should be treated \textbf{Perturbatively}. 

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Wilsonian RG and Pionless NEFT

Nontrivial fixed point ($^1S_0$ channel)

\[
\frac{d}{dt} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 0 & 1 \\ -2 & -2 & -3 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}
\]

\[\nu_1 = +1 : u_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \nu_2 = -1 : u_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \nu_3 = -2 : u_3 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}\]

- \(u_1\) is Relevant \(\Rightarrow\) no suppression \(\Rightarrow\) should be treated Nonperturbatively.

- Others are Irrelevant \(\Rightarrow\) suppression \(\Rightarrow\) should be treated Perturbatively.
Wilsonian RG and Pionless NEFT

RG flows near the fixed point are very similar.

⇒ **Same Power Counting** in $^1S_0$ and $^3S_1 - ^3D_1$ channels.

Where do the physical differences of $^1S_0$ and $^3S_1 - ^3D_1$ channels come from? ($^3S_1 - ^3D_1$: Deuteron, $^1S_0$: No bound state)
Wilsonian RG and Pionless NEFT

RG Phase and Bound States

Off-shell NN scattering amplitude (four-nucleon Green’s function)

\[
-i\mathcal{A}(p_0, p_2^2, p_1^2) = -iV(p_0, p_2^2, p_1^2) + \int_{\Lambda} \frac{d^3k}{(2\pi)^3} (-iV(p_0, p_2^2, k^2)) \frac{i}{p^0 - \frac{k^2}{2M} + i\epsilon} (-i\mathcal{A}(p_0, k^2, p_1^2))
\]
Wilsonian RG and Pionless NEFT

RG Equation (from the amplitude)

\[ \frac{d}{d\Lambda} A = 0 \] (with expansion in powers of momenta)

Same fixed points as the ones obtained from Legendre flow equation (RG equations are different)

\[ (x^*, y^*, z^*) = (0, 0, 0), (-1, -\frac{1}{2}, \frac{1}{2}) \]

Solution near the Nontrivial fixed point,

\[
\begin{pmatrix}
\delta x \\
\delta y \\
\delta z
\end{pmatrix}
= a \begin{pmatrix}
2 \\
1 \\
-4
\end{pmatrix}
\left( \frac{\Lambda}{\Lambda_0} \right)^2
+ b \begin{pmatrix}
0 \\
-1 \\
1
\end{pmatrix}
\left( \frac{\Lambda}{\Lambda_0} \right)
+ c \begin{pmatrix}
1 \\
1 \\
-1
\end{pmatrix}
\left( \frac{\Lambda_0}{\Lambda} \right)
\]

Relevant
Renormalized Amplitude

Substitute the solution into $\mathcal{A}$ (theory near the nontrivial fixed point)

$$\left(\mathcal{A}(p_0, p'^2, p^2)\right)^{-1} = -\frac{M}{4\pi} \left[ \frac{2}{\pi} c \Lambda_0 - \frac{4}{\pi} \frac{M p_0}{\Lambda_0} - i \sqrt{M p_0} \right] + \cdots$$

\cdots denotes contributions higher order in $1/\Lambda_0$. 
Wilsonian RG and Pionless NEFT

**Analytic Structure and Bound State**

\[ c < 0 \text{ ("Strong coupling phase") } \]

A physical pole do exist

\[ p_0 \approx -\frac{1}{M} \left( \frac{2}{\pi} c \Lambda_0 \right)^2 \left[ 1 + \frac{16}{\pi^2} bc \right] \]

\[ c > 0 \text{ ("Weak coupling phase") } \]

A physical pole does not exist

- \( ^1S_0 \) channel is in "**Weak coupling phase**"

- \( ^3S_1 - ^3D_1 \) channel is in "**Strong coupling phase**"
• We have proposed the way to determine a **Power Counting** for EFT by quantum theoretical Dimensional Analysis using Wilsonian RG. Pionless NEFT as a simplest example.

• We have found two RG phases in Pionless NEFT, and given them physical interpretations.
Future Problems

Pionfull NEFT

Our Expectations: Due to the difference of pion interactions in each channel, the RG flow is driven to

- Weak coupling phase in $^1S_0$ channel
- Strong coupling phase in $^3S_1 - ^3D_1$ channel

Three-Body

- Nonperturbative renormalization: each perturbative diagram has only weak cutoff dependence. However, the infinite sum of diagrams has strong cutoff dependence.

- Limit cycle behavior of $D_0$, $\mathcal{L}_3 = \frac{D_0}{M\Lambda_0^4} (N^\dagger N)^3$
Thank you for listening