Charge quantization conditions based on the Atiyah-Singer index theorem

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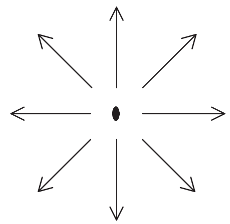
Contents

References

- S. Deguchi and K. Kitsukawa, PTP 115 (2006) 1137, hep-th/0512063.
- S. Deguchi, in preparation.

1. Introduction

The magnetic field due to a point magnetic monopole of strength g situated at the origin is given by



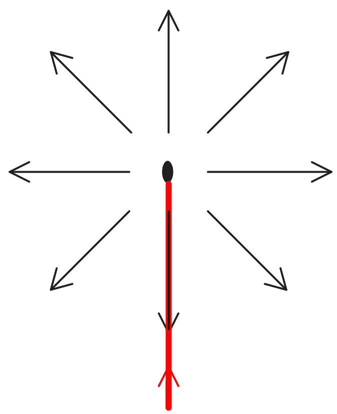
$$\boldsymbol{B}_g = g\frac{\boldsymbol{r}}{r^3}\,, \quad g : \text{magnetic charge}\,.$$

One of the vector potentials that yield $\boldsymbol{B}_g$, with the relation $\boldsymbol{B}_g = \nabla \times \boldsymbol{A}$, is found to be

$$\boldsymbol{A}_g = \frac{g(1-\cos\theta)}{r\sin\theta}\,\boldsymbol{e}_\phi\,, \qquad \begin{cases} \theta : \text{zenith angle},\ 0 \le \theta \le \pi \\ \phi : \text{azimuthal angle},\ 0 \le \phi < 2\pi \\ \boldsymbol{e}_\phi : \text{unite vector in the } \phi\text{-direction} \end{cases}$$

This potential has singularities:

- $r = 0$: monopole singularity
- $\theta = \pi$: Dirac string singularity



Existence of the singularities guarantees

$$\nabla \cdot \boldsymbol{B}_g = \nabla \cdot (\nabla \times \boldsymbol{A}_g)(\neq 0) = 4\pi g\delta^3(\boldsymbol{r})\,.$$

Now, we consider **quantum mechanics** for a particle of electric charge e in the monopole background. Then we find the Dirac quantization condition (Dirac,1931),

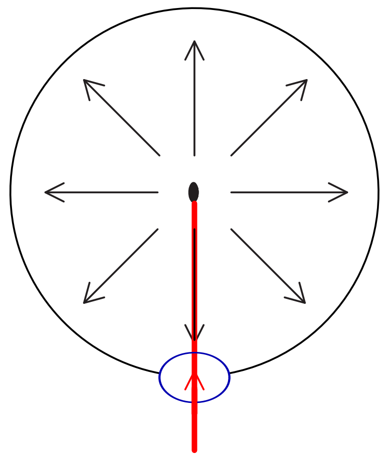
$$eg = \frac{n}{2}\,, \quad n = 0,\, \pm 1,\, \pm 2, \ldots \quad (\text{in natural unites } c = \hbar = 1)\,.$$

Dirac found this condition in the following way: The Schrödinger equation and its solution are given by

$$-\frac{1}{2m}\left(\nabla - ie\boldsymbol{A}_g\right)^2\psi = E\psi \;\Rightarrow\; \psi = \psi_0 \exp\left[ie\int_C d\boldsymbol{r}\cdot\boldsymbol{A}_g\right], \quad \psi_0 : \text{wave function of a free particle}$$

The phase of the wave function can change modulo 2π under a single turn of the wave function around the Dirac string,

$$e\oint_C d\boldsymbol{r}\cdot\boldsymbol{A}_g = 2\pi n\,.$$



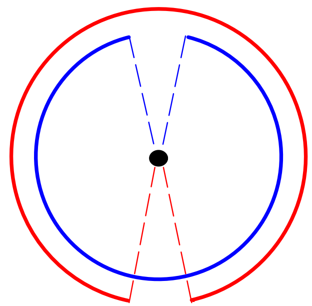
By taking C to be an extremely small loop and using Stokes' theorem, the integral in the LHS reduces to the total magnetic flux due to g; thus, $e \times 4\pi g = 2\pi n$.

Without treating the Dirac string, Wu and Yang derived $eg = n/2$ using two gauge potentials (Wu and Yang,1975).

Consider two potentials that give the magnetic field $\boldsymbol{B}_g$,

$$\boldsymbol{A}_{\mathrm{N}} = \frac{g(1-\cos\theta)}{r\sin\theta}\,\boldsymbol{e}_\phi \quad \text{for} \quad \theta < \pi - \epsilon : \; U_{\mathrm{N}}$$

$$\boldsymbol{A}_{\mathrm{S}} = \frac{g(-1-\cos\theta)}{r\sin\theta}\,\boldsymbol{e}_\phi \quad \text{for} \quad \theta > \epsilon : \; U_{\mathrm{S}}\,.$$



The potential $\boldsymbol{A}_{\mathrm{N}}$ is regular on the region U_{N}, while $\boldsymbol{A}_{\mathrm{S}}$ is regular on U_{S}. No string singularities in this system.

In the overlap region $U_{\mathrm{N}} \cap U_{\mathrm{S}}$, the following gauge transformation is valid:

$$\psi_{\mathrm{N}} = \exp\left[ie\int_C d\boldsymbol{r}\cdot(\boldsymbol{A}_{\mathrm{N}} - \boldsymbol{A}_{\mathrm{S}})\right]\psi_{\mathrm{S}} = \exp\left[ie\int_C d\boldsymbol{r}\cdot\nabla(2g\phi)\right]\psi_{\mathrm{S}} = e^{2ieg\phi}\psi_{\mathrm{S}}\,,$$

where ψ_{N} and ψ_{S} are wave functions on U_{N} and U_{S}, respectively. Comparing $\psi_{\mathrm{N}}(2\pi) = e^{4\pi ieg}\psi_{\mathrm{S}}(2\pi)$ with $\psi_{\mathrm{N}}(0) = \psi_{\mathrm{S}}(0)$ leads to $eg = n/2$.

In Dirac's method and Wu-Yang's method, $eg = n/2$ is derived from consideration of the Dirac phase factor $\exp\left[ie\int d\boldsymbol{r}\cdot\boldsymbol{A}\right]$.

In addition to $eg = n/2$, the Schwinger quantization condition (Schwinger,1966) is known,

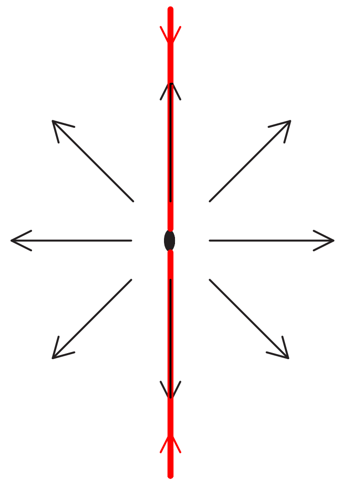
$$eg = n\,, \quad n = 0, \pm 1, \pm 2, \cdots \quad (\text{in natural unites } c = \hbar = 1)\,.$$

Schwinger discovered this condition in study of a relativistic quantum field theory of electric and magnetic charges (J. Schwinger, Phys. Rev. 144 (1966), 1087). There, it was verified that the relativistic invariance at the *operator level* is maintained only when the gauge potential involves an infinite string singularity and $eg = n$ is satified.

A suitable potential that gives $\boldsymbol{B}_g$ is

$$\boldsymbol{A}_{\text{Schwinger}} = \frac{g(-\cos\theta)}{r\sin\theta}\,\boldsymbol{e}_\phi\,.$$

This potential has singularities at both the north and south poles.



In the present work,

- We derive the charge quantization conditions $eg = n/2$ and $eg = n$ by utilizing the Atiyah-Singer index theorem in two dimensions.

- We treat the Dirac potentials $\boldsymbol{A}_{\mathrm{N}}$, $\boldsymbol{A}_{\mathrm{S}}$ and the Schwinger potential $\boldsymbol{A}_{\mathrm{Schwinger}}$ in a unified manner. This can be done by taking

$$\boldsymbol{A}_\kappa = \frac{g(\kappa - \cos\theta)}{r\sin\theta}\,\boldsymbol{e}_\phi\,, \qquad \boldsymbol{A}_\kappa = \begin{cases} \boldsymbol{A}_{\mathrm{N}} & \text{for } \kappa = 1 \\ \boldsymbol{A}_{\mathrm{S}} & \text{for } \kappa = -1 \\ \boldsymbol{A}_{\mathrm{Schwinger}} & \text{for } \kappa = 0 \end{cases}$$

2. Atiyah-Singer index theorem in two dimensions (Atiyah and Singer, 1968)

Let $\mathcal{M}$ be a two-dimensional compact manifold. In terms of local coordinates (q^α) $(\alpha = 1, 2)$ on $\mathcal{M}$, the Dirac operator is expressed as

$$i\not{D} \equiv i\sigma_a e_a{}^\alpha D_\alpha \;\; (a, \alpha = 1, 2), \quad D_\alpha \equiv \frac{\partial}{\partial q^\alpha} + \frac{i}{2}\omega_\alpha \sigma_3 - ieA_\alpha \,.$$

Here σ_1, σ_2, σ_3 are the Pauli matrices, $e_a{}^\alpha$ is an inverse zweibein on $\mathcal{M}$, ω_α is a spin connection in two dimensions, A_α is a Yang-Mills field, and e is a coupling constant.

Consider the positive chirality zero-modes $\varphi^+_{\nu_+}$ $(\nu_+ = 1, \ldots, \mathfrak{n}_+)$ and the negative chirality zero-modes $\varphi^-_{\nu_-}$ $(\nu_- = 1, \ldots, \mathfrak{n}_-)$ of $i\not{D}$, characterized by

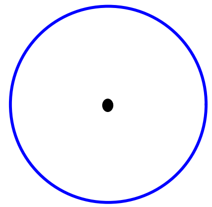
$$i\not{D}\varphi^\pm_{\nu_\pm} = 0 \,, \qquad \sigma_3\varphi^\pm_{\nu_\pm} = \pm\varphi^\pm_{\nu_\pm} \,,$$

where $\mathfrak{n}_+$ $(\mathfrak{n}_-)$ denotes the number of positive (negative) chirality zero-modes. Then, the Atiyah-Singer index theorem in two dimensions reads

$$\mathfrak{n}_+ - \mathfrak{n}_- = \frac{e}{4\pi}\int_{\mathcal{M}} d^2q \, \mathrm{tr}\, \varepsilon^{\alpha\beta} F_{\alpha\beta}$$

where $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha - ie[A_\alpha, A_\beta]$, and tr is the trace over the gauge group.

Now, we consider the case in which $\mathcal{M} = S^2$ and gauge group$= U(1)$. Then the Atiyah-Singer index theorem reads, in the coordinates $(q^1, q^2) = (\theta, \phi)$,



$$\mathfrak{n}_+ - \mathfrak{n}_- = \frac{e}{4\pi} \int_{S^2} d\theta d\phi \, \varepsilon^{\alpha\beta} F_{\alpha\beta} \,, \qquad F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha \,.$$

Also, we choose the monopole gauge potential $\boldsymbol{A}_\kappa$; in a component form, it is written as

$$A_a = \delta_{a2} \frac{g(\kappa - \cos\theta)}{r \sin\theta} \qquad \text{in the local orthonormal frame}\,,$$
$$A_\alpha = e_\alpha{}^a A_a = \delta_{\alpha 2} g(\kappa - \cos\theta) \qquad \text{in the general coordinates}\,.$$

Accordingly, it follows that $F_{ab} = \epsilon_{ab} \dfrac{g}{r^2}$ and $F_{\alpha\beta} = \epsilon_{\alpha\beta}\, g \sin\theta$. The Atiyah-Singer index theorem reduces to

$$\boxed{\mathfrak{n}_+ - \mathfrak{n}_- = 2eg}$$

This seems to be a charge quantization condition. *But*, at this stage, we don't know what numbers the LHS may take: all integers or even numbers or some particular numbers? In order to know possible numbers in the LHS, we need to solve $i\not{D}\varphi = 0$.

3. Solving the massless Dirac equation

The massless Dirac equation $i\not{D}\varphi = 0$ can be written in a matrix form,

$$\begin{pmatrix} 0 & \nabla_\theta - \frac{i}{\sin\theta}\nabla_\phi \\ \nabla_\theta + \frac{i}{\sin\theta}\nabla_\phi & 0 \end{pmatrix} \begin{pmatrix} u^+(\theta,\phi) \\ u^-(\theta,\phi) \end{pmatrix} = 0\,, \qquad \varphi \equiv \begin{pmatrix} u^+ \\ u^- \end{pmatrix},$$

where

$$\nabla_\theta \equiv \frac{\partial}{\partial\theta} + \frac{1}{2}\cot\theta\,, \qquad \nabla_\phi \equiv \frac{\partial}{\partial\phi} - ieg(\kappa - \cos\theta)\,.$$

Because of the periodicity in ϕ, $u^\pm$ takes the form $u^\pm(\theta,\phi) = v^\pm(\theta)\exp(im_\pm\phi)$. Here m_+ and m_- are half-integers, that is, $m_+\,,\, m_- = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots,$ because the spinor field φ has to change sign under a 2π rotation in ϕ. The differential equation in θ is obtained as

$$\left[\frac{d}{d\theta} + \left(\frac{1}{2} \mp eg\right)\cot\theta \mp \frac{m_\pm - eg\kappa}{\sin\theta}\right] v^\pm(\theta) = 0\,.$$

The solutions of this equation are readily found to be

$$v^\pm_{m_\pm}(\theta) = \left(\sin\frac{\theta}{2}\right)^{s^\pm_{m_\pm}} \left(\cos\frac{\theta}{2}\right)^{c^\pm_{m_\pm}}.$$

Here the constants $s^{\pm}_{m_\pm}$ and $c^{\pm}_{m_\pm}$ are defined by

$$s^{\pm}_{m_\pm} \equiv \pm\{m_\pm - eg(\kappa - 1)\} - \frac{1}{2}\,, \qquad c^{\pm}_{m_\pm} \equiv \mp\{m_\pm - eg(\kappa + 1)\} - \frac{1}{2}\,.$$

The solution $v^{\pm}_{m_\pm}(\theta)$ diverges at neither $\theta = 0$ nor π, if and only if $s^{\pm}_{m_\pm}, c^{\pm}_{m_\pm} \geq 0$.

- The conditions $s^{+}_{m_+}$, $c^{+}_{m_+} \geq 0$ can be expressed as

$$\frac{1}{2} + eg(\kappa - 1) \leq m_+ \leq -\frac{1}{2} + eg(\kappa + 1) \quad \Rightarrow \quad eg \geq \frac{1}{2}\,.$$

- The conditions $s^{-}_{m_-}$, $c^{-}_{m_-} \geq 0$ can be expressed as

$$\frac{1}{2} + eg(\kappa + 1) \leq m_- \leq -\frac{1}{2} + eg(\kappa - 1) \quad \Rightarrow \quad eg \leq -\frac{1}{2}\,.$$

The conditions $s^{+}_{m_+}, c^{+}_{m_+} \geq 0$ and the conditions $s^{-}_{m_-}, c^{-}_{m_-} \geq 0$ are never satisfied simultaneously with a given eg. The possible solutions of $i\not{D}\varphi = 0$ are restricted to

$$\varphi^{+}_{m_+} = \begin{pmatrix} u^{+}_{m_+} \\ 0 \end{pmatrix} \text{ for } eg \geq \frac{1}{2}\,, \quad \varphi^{\pm}_{m_\pm} = 0 \text{ for } |eg| < \frac{1}{2}\,, \quad \varphi^{-}_{m_-} = \begin{pmatrix} 0 \\ u^{-}_{m_-} \end{pmatrix} \text{ for } eg \leq -\frac{1}{2}\,,$$

where $u^{\pm}_{m_\pm} = v^{\pm}_{m_\pm}(\theta)\exp(im_\pm\phi)$. The chirality condition $\sigma_3\varphi^{\pm}_{m_\pm} = \pm\varphi^{\pm}_{m_\pm}$ is satisfied.

4. Count of zero-modes and derivation of charge quantization conditions

First, consider the case $\boldsymbol{\kappa = 1}$. The inequality for m_+ and that for m_- read

$$\frac{1}{2} \le m_+ \le -\frac{1}{2} + 2eg\,, \qquad \frac{1}{2} + 2eg \le m_- \le -\frac{1}{2}\,.$$

Suppose that $eg \left(\ge \frac{1}{2} \right)$ is in the interval $\frac{n}{2} \le eg < \frac{n+1}{2} \;\; (n = 1, 2, \ldots)$. Because m_+ takes half-integer values, the allowed values of m_+ are seen to be

$$m_+ = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \frac{2n-1}{2} \quad \Rightarrow \quad \mathfrak{n}_+ \equiv \sharp\big(\varphi^+_{m_+}\big) = n\,, \quad \text{while} \;\; \mathfrak{n}_- = 0\,.$$

Next suppose that $eg \left(\le -\frac{1}{2} \right)$ is in the interval $-\frac{n+1}{2} < eg \le -\frac{n}{2} \;\; (n = 1, 2, \ldots)$. Because m_- also takes half-integer values, the allowed values of m_- are seen to be

$$m_- = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots, -\frac{2n-1}{2} \quad \Rightarrow \quad \mathfrak{n}_- \equiv \sharp\big(\varphi^-_{m_-}\big) = n\,, \quad \text{while} \;\; \mathfrak{n}_+ = 0\,.$$

Therefore the AS index theorem $\mathfrak{n}_+ - \mathfrak{n}_- = 2eg$ gives

$$n = 2eg \;\; \text{for} \;\; eg \ge \frac{1}{2}\,, \qquad 0 = 2eg \;\; \text{for} \;\; |eg| < \frac{1}{2}\,, \qquad -n = 2eg \;\; \text{for} \;\; eg \le -\frac{1}{2}\,.$$

The three relations obtained here,

$$n = 2eg \;\text{ for }\; eg \geq \frac{1}{2}\,, \qquad 0 = 2eg \;\text{ for }\; |eg| < \frac{1}{2}\,, \qquad -n = 2eg \;\text{ for }\; eg \leq -\frac{1}{2}\,,$$

are brought together in the form

$$eg = \frac{n}{2}\,, \quad n = 0,\, \pm 1,\, \pm 2, \ldots\,.$$

This is precisely the Dirac quantization condition.

In the case $\boldsymbol{\kappa = -1}$, we have the Dirac quantization condition *again*. This is quite natural, because the case $\kappa = -1$ is a mirror image of the case $\kappa = 1$.

Finally, consider the case $\boldsymbol{\kappa = 0}$. The inequality for m_+ and that for m_- read

$$\frac{1}{2} - eg \leq m_+ \leq -\frac{1}{2} + eg \,, \qquad \frac{1}{2} + eg \leq m_- \leq -\frac{1}{2} - eg \,.$$

When eg is in the interval $\frac{1}{2} \leq eg < 1$, there are no allowed values of m_+. When $eg \; (\geq 1)$ is in the interval $n \leq eg < n+1 \;\; (n = 1, 2, \ldots)$, the allowed values of m_+ are found to be

$$m_+ = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots, \pm\frac{2n-1}{2} \quad \Rightarrow \quad \mathfrak{n}_+ \equiv \sharp(\varphi^+_{m_+}) = 2n \,, \quad \text{while } \; \mathfrak{n}_- = 0 \,.$$

When eg is in the interval $-1 < eg \leq -\frac{1}{2}$, there are no allowed values of m_-. When $eg \; (\leq -1)$ is in the interval $-(n+1) < eg \leq -n \;\; (n = 1, 2, \ldots)$, the allowed values of m_- are found to be

$$m_- = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots, \pm\frac{2n-1}{2} \quad \Rightarrow \quad \mathfrak{n}_- \equiv \sharp(\varphi^-_{m_-}) = 2n \,, \quad \text{while } \; \mathfrak{n}_+ = 0 \,.$$

Therefore, in this case, the AS index theorem $\mathfrak{n}_+ - \mathfrak{n}_- = 2eg$ gives

$$2n = 2eg \;\text{ for }\; eg \geq 1 \,, \qquad 0 = 2eg \;\text{ for }\; |eg| < 1 \,, \qquad -2n = 2eg \;\text{ for }\; eg \leq -1 \,.$$

The three relations obtained here,

$$2n = 2eg \;\; \text{for} \;\; eg \geq 1\,, \qquad 0 = 2eg \;\; \text{for} \;\; |eg| < 1\,, \qquad -2n = 2eg \;\; \text{for} \;\; eg \leq -1\,,$$

are brought together in the form

$$eg = n\,, \quad n = 0,\, \pm 1,\, \pm 2, \ldots\,.$$

This is precisely the Schwinger quantization condition.

5. Atiyah-Singer Index theorem in the Yang-Mills-Higgs system

Let A_α be a Yang-Mills field on the two-dimensional manifold $\mathcal{M}$ and let $\varPhi$ be an adjoint scalar field on $\mathcal{M}$. The gauge group is now assumed to be $SU(2)$. The fields A_α and $\varPhi$ are thus expanded as

$$A_\alpha = A_\alpha^i \tau_i + A_\alpha^3 \tau_3 \quad (i = 1, 2)\,, \qquad \varPhi = \phi^i \tau_i + \phi^3 \tau_3\,,$$

where τ_1, τ_2, τ_3 are the Pauli matrices. We now impose the normalization condition $\mathrm{tr}(\varPhi^2) = 2$, or equivalently $(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = 1$. Then, $\varPhi$ can be diagonalized as

$$v^\dagger \varPhi v = \tau_3 \quad \Rightarrow \quad \varPhi = v \tau_3 v^\dagger,$$

with a 2×2 unitary matrix $v\,(\in SU(2))$. Using A_α, $\varPhi$ and $\varPsi^i \equiv v \tau_i v^\dagger$, we define the vector field

$$A_\alpha^\perp \equiv A_\alpha - \frac{1}{2e} \epsilon_{ij3} \mathrm{tr}(\varPsi^i D_\alpha \varPhi) \varPsi^j,$$

where $D_\alpha \varPhi \equiv \partial_\alpha \varPhi - i\frac{e}{2}[A_\alpha, \varPhi]$. Furthermore we define the Dirac operator

$$i \not{D}^\perp \equiv i \sigma_a e_a{}^\alpha D_\alpha^\perp\,, \qquad D_\alpha^\perp \equiv \frac{\partial}{\partial q^\alpha} + \frac{i}{2}\omega_\alpha \sigma_3 - i\frac{e}{2} A_\alpha^\perp\,.$$

Consider the chirality zero-modes $\varphi^{t,s}_{\nu_{t,s}}$ $(t, s = +,-\ ;\ \nu_{t,s} = 1, \ldots, \mathfrak{n}_{t,s})$ of $i\not{D}^{\perp}$, characterized by

$$i\not{D}^{\perp}\varphi^{t,s}_{\nu_{t,s}} = 0\,, \qquad (\Phi\otimes\sigma_3)\varphi^{t,s}_{\nu_{t,s}} = ts\,\varphi^{t,s}_{\nu_{t,s}}\,. \qquad \begin{cases} t : \text{eigenvalue of } \Phi \\ s : \text{eigenvalue of } \sigma_3 \end{cases}$$

Here $\mathfrak{n}_{t,s}$ denotes the number of chirality zero-modes specified by (t,s).

We can prove the AS index theorem in the 2-dimensional YMH system, i.e.

$$\mathfrak{n}_{++} - \mathfrak{n}_{+-} - \mathfrak{n}_{-+} + \mathfrak{n}_{--} = \frac{e}{4\pi}\int_{\mathcal{M}} d^2q\,\epsilon^{\alpha\beta}\mathcal{F}_{\alpha\beta}\,,$$

with

$$\mathcal{F}_{\alpha\beta} \equiv \frac{1}{2}\,\mathrm{tr}\left[\Phi F_{\alpha\beta} + \frac{i}{2e}\Phi\big(D_\alpha\Phi D_\beta\Phi - D_\beta\Phi D_\alpha\Phi\big)\right],$$

where $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha - i\dfrac{e}{2}[A_\alpha, A_\beta]\,,$ and tr is the trace over the gauge group.

Note here that $\mathcal{F}_{\alpha\beta}$ is precisely the 't Hooft tensor in two-dimensions.

In the case of $\mathcal{M} = S^2$, the magnetic charge in the YMH system is defined by

$$g \equiv \frac{1}{8\pi} \int_{S^2} d^2q \, \epsilon^{\alpha\beta} \mathcal{F}_{\alpha\beta} = \frac{1}{4\pi} \int_{S^2} \mathcal{F} \, .$$

With this g, the AS index theorem in the YMH system can be expressed as

$$\boxed{\mathfrak{n}_{++} - \mathfrak{n}_{+-} - \mathfrak{n}_{-+} + \mathfrak{n}_{--} = 2eg}$$

In the case of $\mathcal{M} = S^2$, there is some evidence that $\mathfrak{n}_{++} = \mathfrak{n}_{--}$ and $\mathfrak{n}_{+-} = \mathfrak{n}_{-+}$. If it is justified, the AS index theorem takes the form

$$\mathfrak{n}_{++} - \mathfrak{n}_{+-} = eg \, .$$

According to an analysis made by Arafune, Freund and Goebel in 1974, which is based on the *homotopy* theory, the charge quantization in the YMH system is determined to be

$$eg = n \, , \quad n = 0, \pm 1, \pm 2, \ldots \, .$$

The AS index theorem discussed here is compatible with this charge quantization.

6. Summary

- We have derived *both* the charge quantization conditions $eg = n/2$ and $eg = n$ by using the Atiyah-Singer index theorem in two dimensions.

- The *difference* between the Dirac and Schwinger quantization conditions simply results from the fact that

$$\sharp(\text{zero-modes in the case } \kappa = 0, \text{ Schwinger formalism})$$
$$= 2 \times \sharp(\text{zero-modes in the case } \kappa = \pm 1, \text{ Dirac formalism})$$

- Our approach requires neither the classical notion of paths around a string singularity nor the concept of patches and sections.

- The charge quantization conditions are regarded as the necessary and sufficient conditions that zero-modes of the Dirac operator exist and the Atiyah-Singer index-theorem in two dimensions is valid.

- We have generalized the Atiyah-Singer index theorem in two dimensions into the Yang-Mills-Higgs system. This theorem appears to be consistent with the charge quantization condition $eg = n$ found by Arafune, Freund and Goebel.