Supersymmetric Gauge Theories with Matters, Toric Geometries and Random Partitions

Yui NOMA

Department of Physics, Graduate School of Science, Osaka University

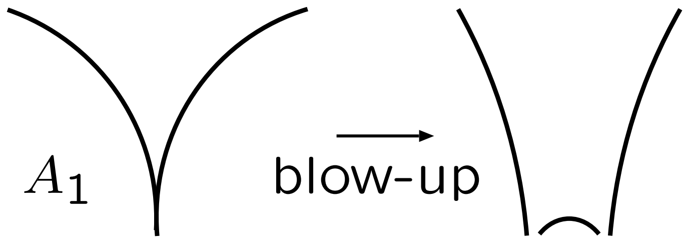
31 Oct @ APS-DPF2006 + JPS2006

This talk is based on [hep-th/0604141] **Y.N.**

Contents

Motivation 1 : **Geometric engineering**

$\mathcal{N}=2$ $SU(2)$ gauge theory $\leftarrow$ Type IIA compactified on C.Y. 3-fold (A_1-singularity)



$W^{\pm}$ $\leftarrow$ D2-brane wrapped on $\mathbb{CP}^1$

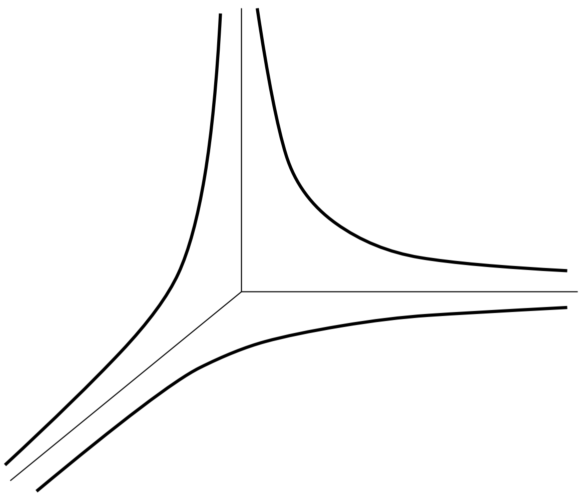
Z $\leftarrow$ 3-form

$\mathbb{R}^4$	C.Y. 3-fold	Type IIA
	$\downarrow$	
$\mathbb{R}^4 \times S^1$	C.Y. 3-fold	M-theory

Motivation 2 : **Melting crystals**

Instanton part of Nekrasov's formula ⟷ topological vertex ↕ melting crystal

↖↘ Random partitions ↙↗



Perturbative part of prepotential ⟷ Geometry ↕ Crystal

↖↘ Ground state of random partitions ↙↗

Correspondences

Perturbative sector ↔ Dimension of a Hilbert space with no matters

(q-deformed) $\mathcal{N} = 2$ gauge theory ⟷ Toric varieties

Nekrasov formula with no matters
↕
Partition function

Polyhedron
↕
Ground state (with no matters)

Statistical models of partitions
(2D CFT)

Free fermions

Perturbative sector ↔ Dimension of a Hilbert space
with a adj. matter

(q-deformed)
$\mathcal{N} = 2$ gauge theory ⟷ Toric varieties

Nekrasov formula
with a adj. matter
↕
Partition function

Polyhedron
↕
Ground state
(with a adj. matter)

Statistical models of partitions
(2D CFT)

Free fermions

(q-deformed)
$\mathcal{N} = 2$ gauge theory ⟷ Toric varieties

Statistical models of partitions
(2D CFT)

4D $\mathcal{N}=2^*$ gauge theory : Nekrasov formula

◦ 4D Nekrasov formula for SU(2) with a massive adj. matter

[Nekrasov and Okounkov '03]

$$Z_{\text{Nek}\,adj} = Z^{pert}_{\text{Nek}\,adj} \sum_{\lambda} z^{4|\lambda^{(1)}|+4|\lambda^{(2)}|} \times \prod_{(r,i)\neq(s,j)} \frac{(a_{rs}/\hbar + \lambda_i^{(s)} - \lambda_j^{(r)} - i + j)\cdot(\frac{m_{adj}+a_{rs}}{\hbar} + +j - i)}{(\frac{a_{rs}}{\hbar} + j - i)\cdot(\frac{m_{adj}+a_{rs}}{\hbar} + \lambda_i^{(s)} - \lambda_j^{(r)} - i + j)},$$

$$\mathcal{F}^{pert}_{adj.} = \lim_{\hbar\to 0} \hbar^2 \ln Z^{pert}_{\text{Nek}\,adj.},$$

$$\mathcal{F}^{inst}_{adj.} = \lim_{\hbar\to 0} \hbar^2 (\ln Z_{\text{Nek}} - \ln Z^{pert}_{\text{Nek}}).$$

$Z_{\text{Nek}\,adj.}$: Nekrasov's partition function

$\hbar$: graviphoton background $\simeq$ "string coupling"

$\lambda^{(r)}$: partitions, $(r=1,2)$,

$|\lambda|$ $:= \sum_{i=1}^{\infty} \lambda_i$

a_{rs} $:= a_r - a_s$, a_s $(r=1,2)$ is the vev. of the scalar $\sum_{r=1}^{2} a_r = 0$,

$|\lambda^1| + |\lambda^2|$: instanton number

5D $\mathcal{N}=1^*$ gauge theory : Nekrasov formula

∘ 5D ($\mathbb{R}^4 \times S^1$) generalized (q-deformed) Nekrasov's formula

$$Z_{\text{Nek}\,5D\,adj} = Z^{pert}_{\text{Nek}\,5D\,adj} \sum_{\lambda} z^{4|\lambda^{(1)}|+4|\lambda^{(2)}|}$$
$$\times \prod_{(r,i)\neq(s,j)} \frac{\left[2(\frac{a_{rs}}{\hbar}+\lambda_i^{(s)}-\lambda_j^{(r)}-i+j)\right]_{q^{1/2}} \cdot \left[2(\frac{\mu}{2}+\frac{a_{rs}}{\hbar}+j-i)\right]_{q^{1/2}}}{\left[2(\frac{a_{rs}}{\hbar}+j-i)\right]_{q^{1/2}} \cdot \left[2(\frac{\mu}{2}+\frac{a_{rs}}{\hbar}+\lambda_i^{(s)}-\lambda_j^{(r)}-i+j)\right]_{q^{1/2}}}$$

$$\mathcal{F}^{pert}_{5D\,adj.} = \lim_{\hbar\to 0} \hbar^2 \ln Z^{pert}_{\text{Nek},5D\,adj.},$$
$$\mathcal{F}^{inst}_{5D\,adj.} = \lim_{\hbar\to 0} \hbar^2 (\ln Z_{\text{Nek},5D\,adj.} - \ln Z^{pert}_{\text{Nek},5D\,adj.}).$$

β : circumference of S^1 in the 5th direction,

$[n]_{q^{1/2}} := \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}$, is called "$q$-integer",

q := $\exp(-\beta\hbar/2)$, deformation parameter.

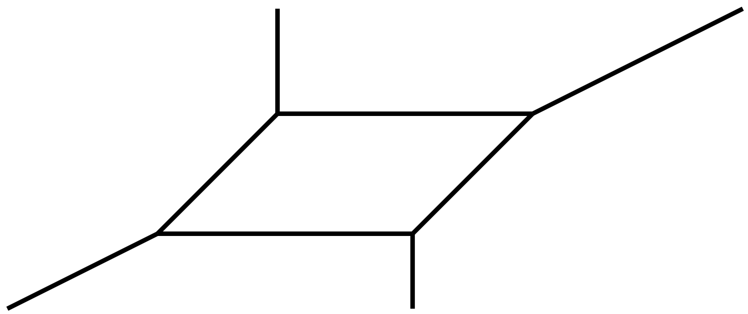
(q-deformed)

$\mathcal{N} = 2$ gauge theory ⟷ Toric varieties

Statistical models of partitions
(2D CFT)

Polyhedron

$X \leftarrow$ ALE space (A_1)
$\downarrow$
$\mathbb{CP}^1$ non cpt. C.Y. 3-fold



Polyhedron $\mathcal{P}$ on a $3D$ lattice M.
$\mathcal{P} \cap M$: holomorphic functions on the variety (crystal)

With no matter.

$$Z^{inst}_{\text{Nek}\,5D} \;\propto\; Z^{top}_X,$$ [Iqbal, Kashani-Poor '03]

[Maeda, Nakatsu, **Y.N.** and Tamakoshi '05]

$$\mathcal{F}^{pert}_{5D} = \lim_{\hbar\to 0} \hbar^{-2}\,\text{"Card"}\,(\mathcal{P}\cap M) + const., \qquad \text{for } \beta \gg 1.$$

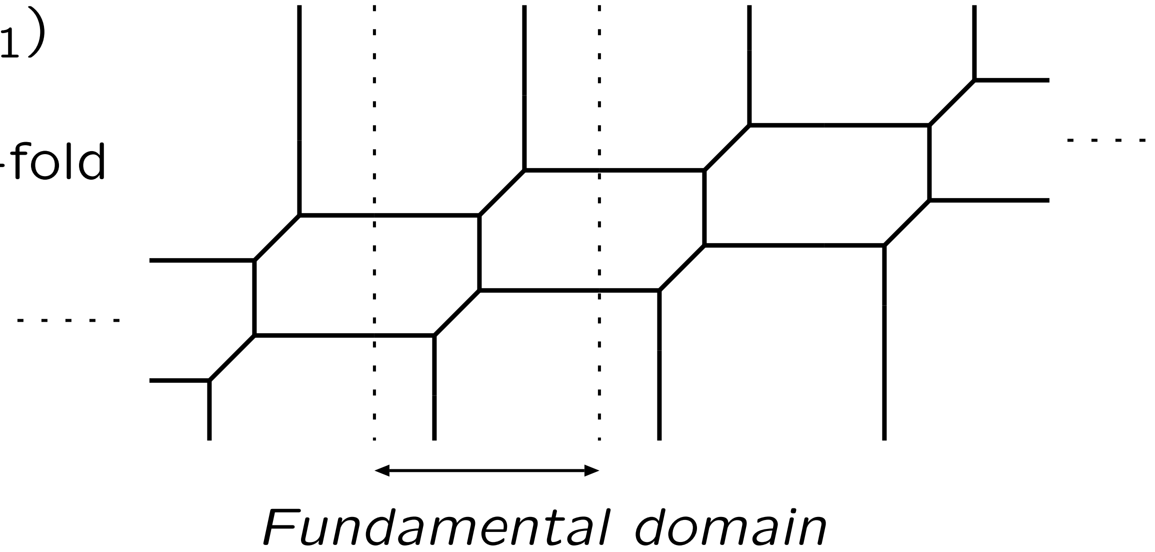
"Card" means it is regularized.

Polyhedron

X_{adj} $\leftarrow$ ALE space (A_1)

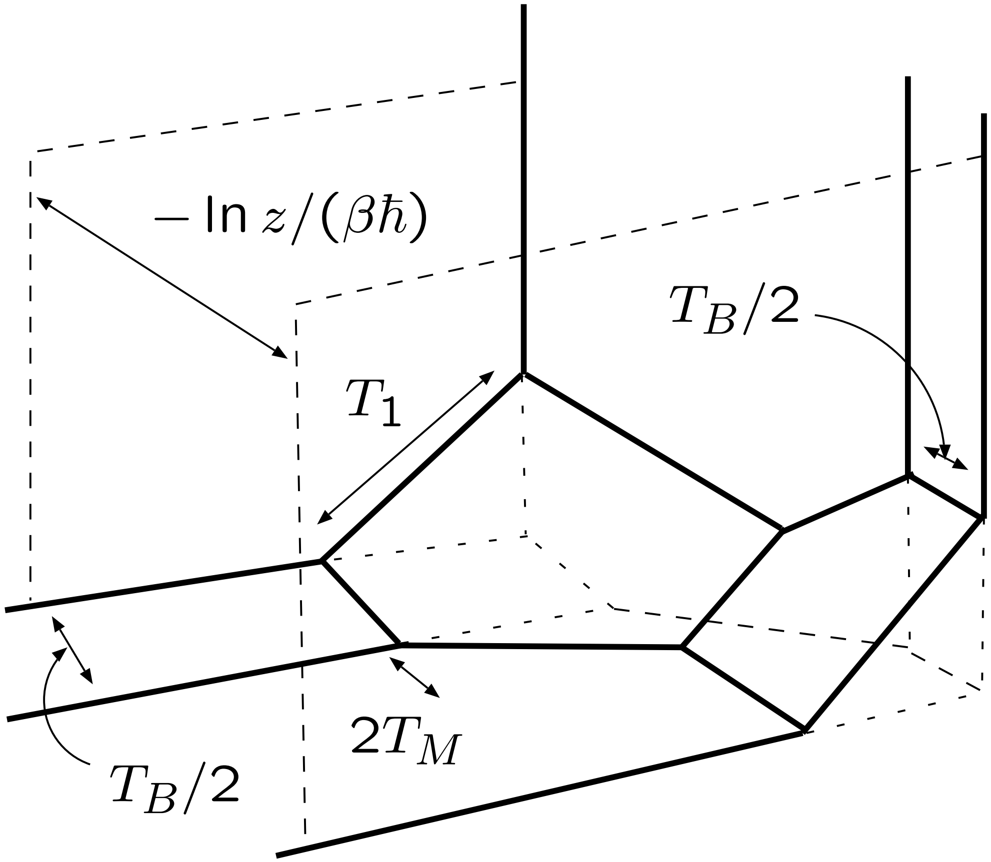
$\downarrow$

"T^2" non cpt. C.Y. 3-fold



This is a polyhedron considered to be periodic.

Polyhedron



Fundamental domain $\mathcal{P}_{adj}$

$\mathcal{P}_{adj} \cap M$: crystal

[Hollowood, Iqbal, Vafa '03]

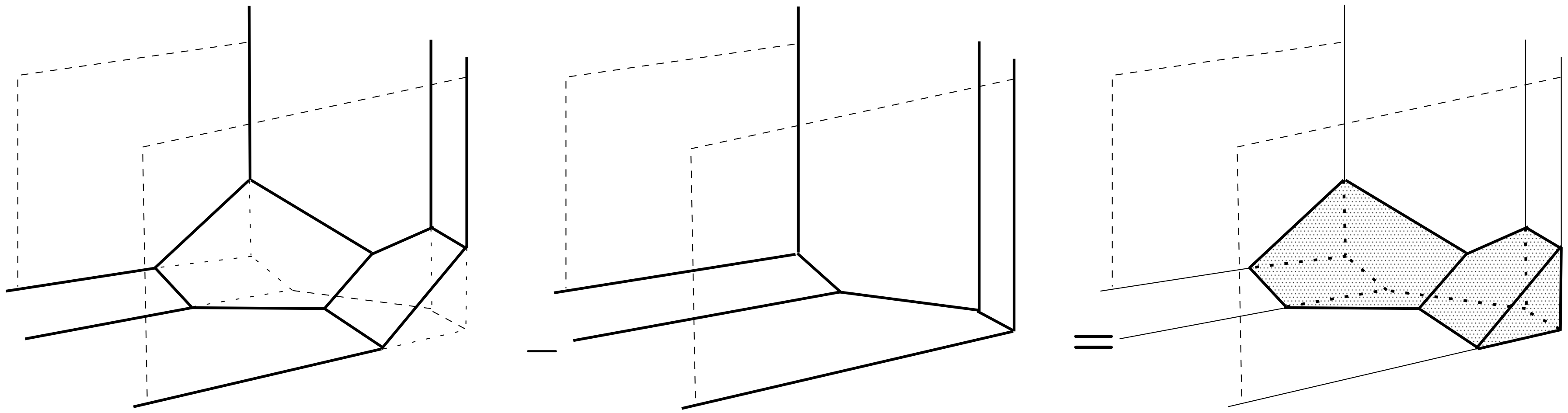
$$Z^{inst}_{\text{Nek}\,5D\,adj} \;\propto\; Z^{top},$$

(q-deformed)
$\mathcal{N} = 2$ gauge theory ⟷ Toric varieties

Statistical models of partitions
(2D CFT)

Prepotential from Polyhedron

We regularize the cardinality.

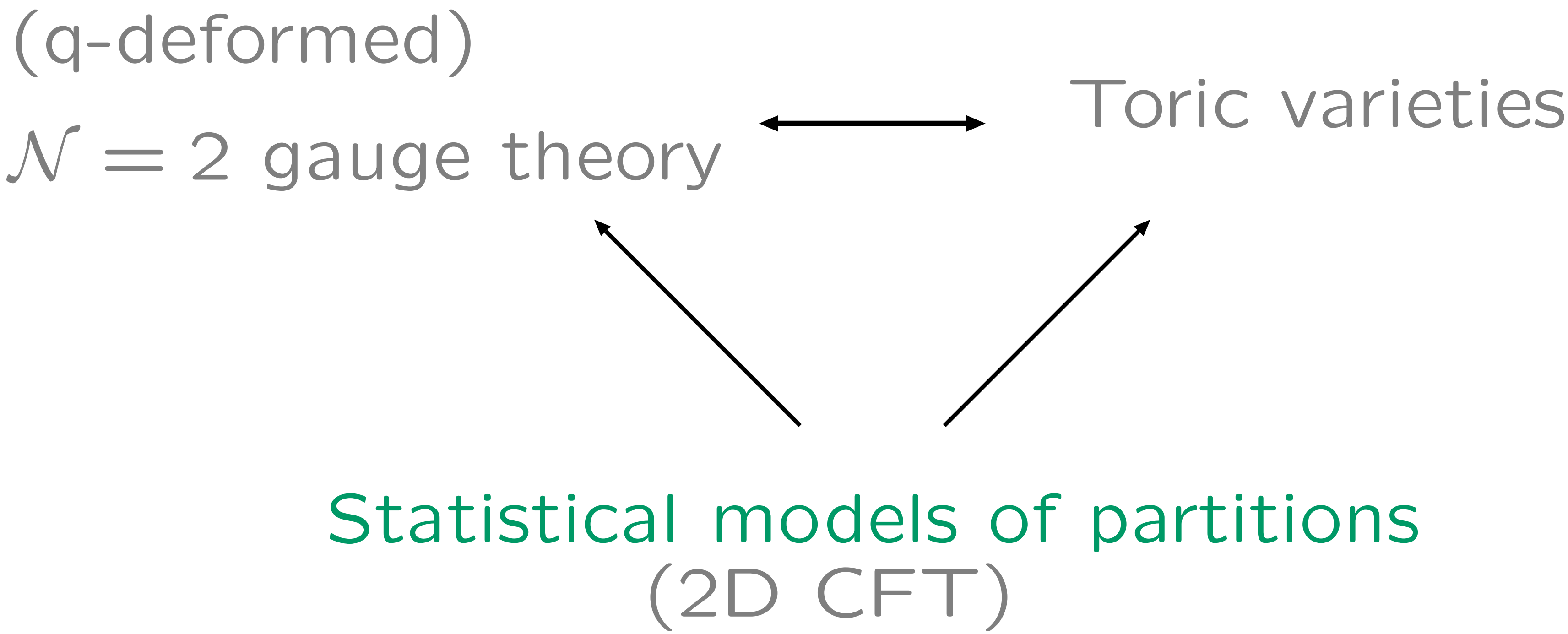


Prepotential emerges from cardinality

$$\mathcal{F}^{pert}_{5D\,adj} = \lim_{\hbar\to 0} \hbar^2 \mathrm{Card}(\mathcal{P}^c_{adj} \cap M) + \mathrm{const.}, \qquad \text{for } \beta \gg 1.$$

$$\begin{aligned} T_1 &= 2a_2, & T_M &= m_{adj}/\hbar, \\ T_B &= -\ln z/(\beta\hbar) - 2T_M, & \ln z &= \exp(-8\pi^2/g^2_{YM}). \end{aligned}$$

$\beta\hbar T_B$, $\beta\hbar T_1$ $\beta\hbar T_M$ are fixed.

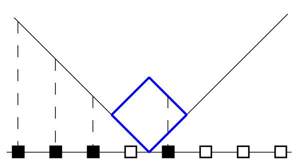

(q-deformed)
$\mathcal{N} = 2$ gauge theory
Toric varieties
Statistical models of partitions
(2D CFT)

Statistical model of partitions : Embedding

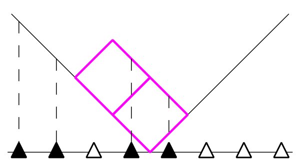
Two partition $\lambda^{(r)}$, $r = 1, 2$ can be embedded to a single partition ν s.t.

$$\left\{x_i(\nu(\lambda^{(r)}, p_r)); i \geq 1\right\} = \bigcup_{r=1}^{2} \left\{2(x_{i_r}(\lambda^{(r)}) + \tilde{p}_r); i_r \geq 1\right\},$$

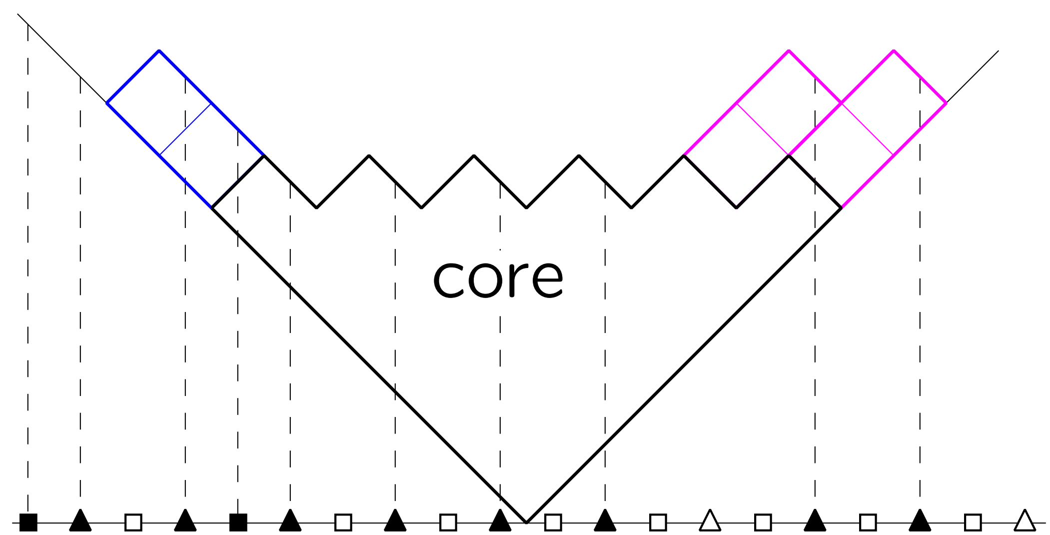
$$\begin{aligned} x_i(\nu) \quad &:= \nu_i - i + \tfrac{1}{2}, \\ \lambda^{(r)} \quad &: r\text{-th partition}, \qquad p_r \quad : \text{charge for } r\text{-th parititon}, \\ \tilde{p}_r \quad &:= p_r + \xi_r, \qquad \xi_r \quad := \tfrac{1}{2}(r - \tfrac{3}{2}). \end{aligned}$$



$\lambda^{(1)} = (1)$, $p_1 = -3$.



$\lambda^{(2)} = (1, 1)$, $p_2 = 3$.



Statistical model of partitions

- π a sequence of partitions *s.t.*

$$\pi(-\mu) \prec \cdots \prec \pi(-2) \prec \pi(-1) \prec \pi(0),$$
$$\pi(0)^t \succ \pi(1)^t \succ \pi(2)^t \succ \cdots \succ \pi(\mu)^t,$$
$$\pi(\mu) = \pi(-\mu).$$

where $\quad \mu \succ \lambda \Leftrightarrow \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \; .$

$$\begin{aligned}
Z_{\mathrm{SP}} &:= \sum_{\pi} \left(\prod_{m=-\mu+1}^{\mu+1} q^{|\pi(m)|} \right) (-zq^{-\mu})^{|\pi(0)|} (-q^{-\mu+1})^{|\pi(\mu)|}. \\
&= \sum_{\lambda} (-zq^{-\mu})^{|\lambda|} \left(\sum_{\substack{\pi \\ \pi(0)=\lambda}} \prod_{m=-\mu+1}^{\mu+1} q^{|\pi(m)|} (-q^{-\mu+1})^{|\pi(\mu)|} \right) \\
&= \sum_{\text{core}} Z^{pert}_{\mathrm{SP}}(p) \cdot Z^{inst}_{\mathrm{SP}}(p),
\end{aligned}$$

where

$$Z^{pert}_{\mathrm{SP}}(p) := \sum_{\substack{\pi \\ \pi(0)=\text{core}}} \left(\prod_{m=-\mu+1}^{\mu+1} q^{|\pi(m)|} \right) \times (-zq^{-\mu})^{|\pi(0)|} (-q^{-\mu+1})^{|\pi(\mu)|}.$$

Statistical model of partitions : Ground state

$$P_{GP}(p) := \{\pi | \pi(0) = \text{core}(p)\}\,.$$

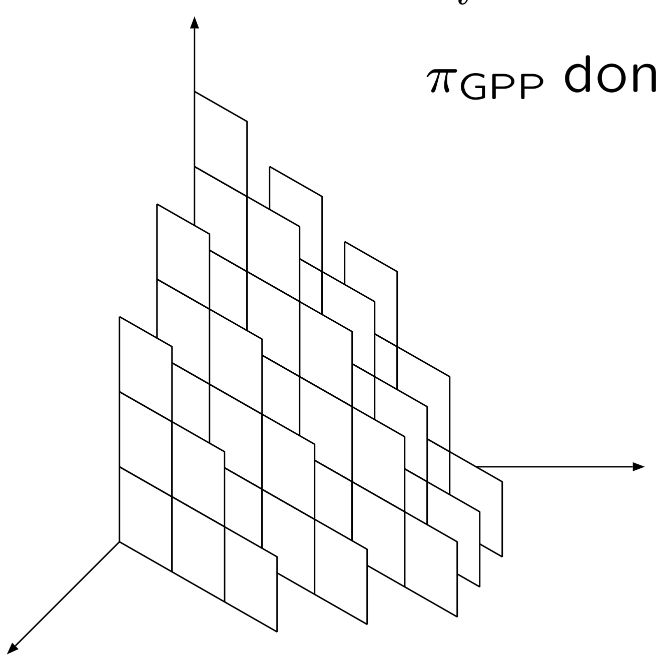
$$^{\exists}\pi_{\text{GPP}} \in P_{GP}(p),\ \ s.t.\ |\pi_{\text{GPP}}| \leq |\pi|,\ \ ^{\forall}\pi \in P_{GP}(p).$$

π_{GPP} : ground state.

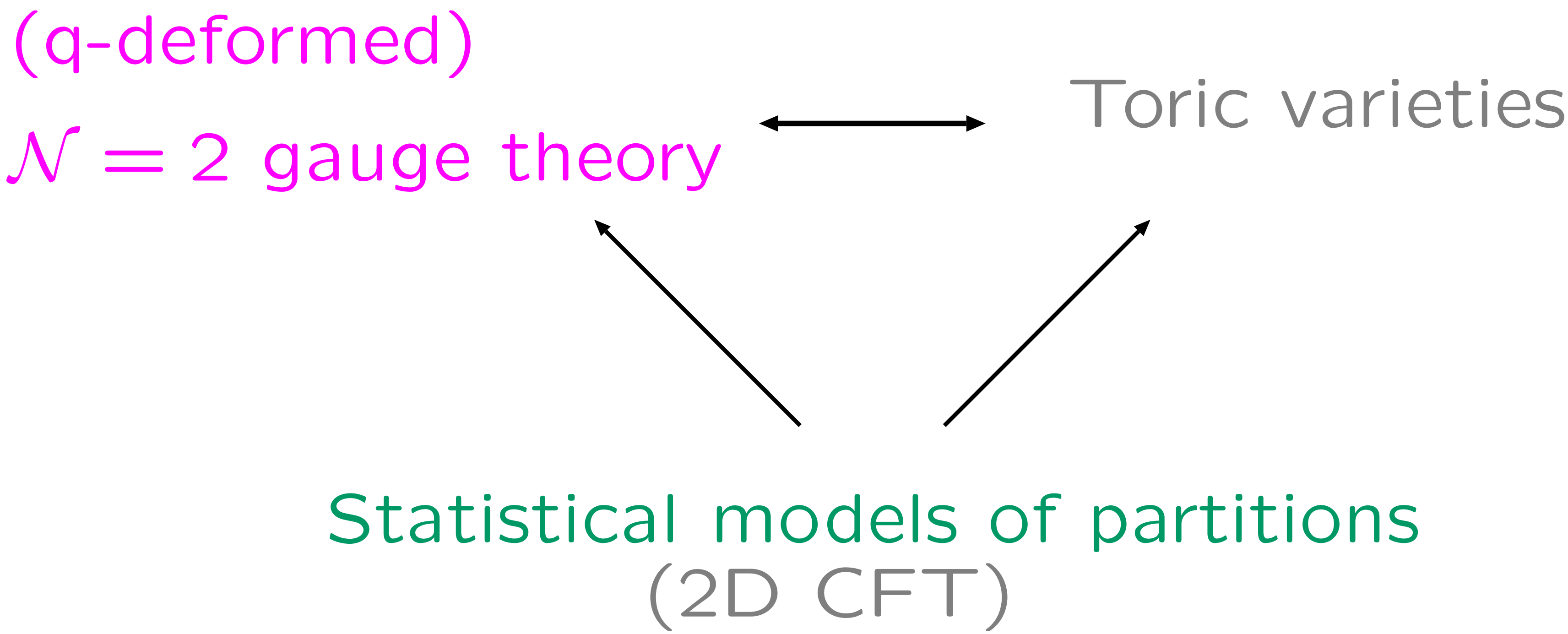
$$\pi_{\text{GPP}\,i}(n) \;=\; \begin{cases} \max\{\text{core}(p)_i - n,\, \lambda_i^{\mu}\} & \text{for } (n \geq 0) \\ \max\{\text{core}(p)_{i+n},\, \lambda_i^{\mu}\} & \text{for } (n < 0), \end{cases}$$

$$\lambda_i^{\mu} : \;=\; \max\{\text{core}(p)_{i+\mu}, \text{core}_i - \mu\}.$$

π_{GPP} donimates $Z_{\text{SP}}^{pert}(p)$ at $q \to 0$ ($\beta \to \infty$).



Example of π_{GPP} in the case of $\mu = 2$

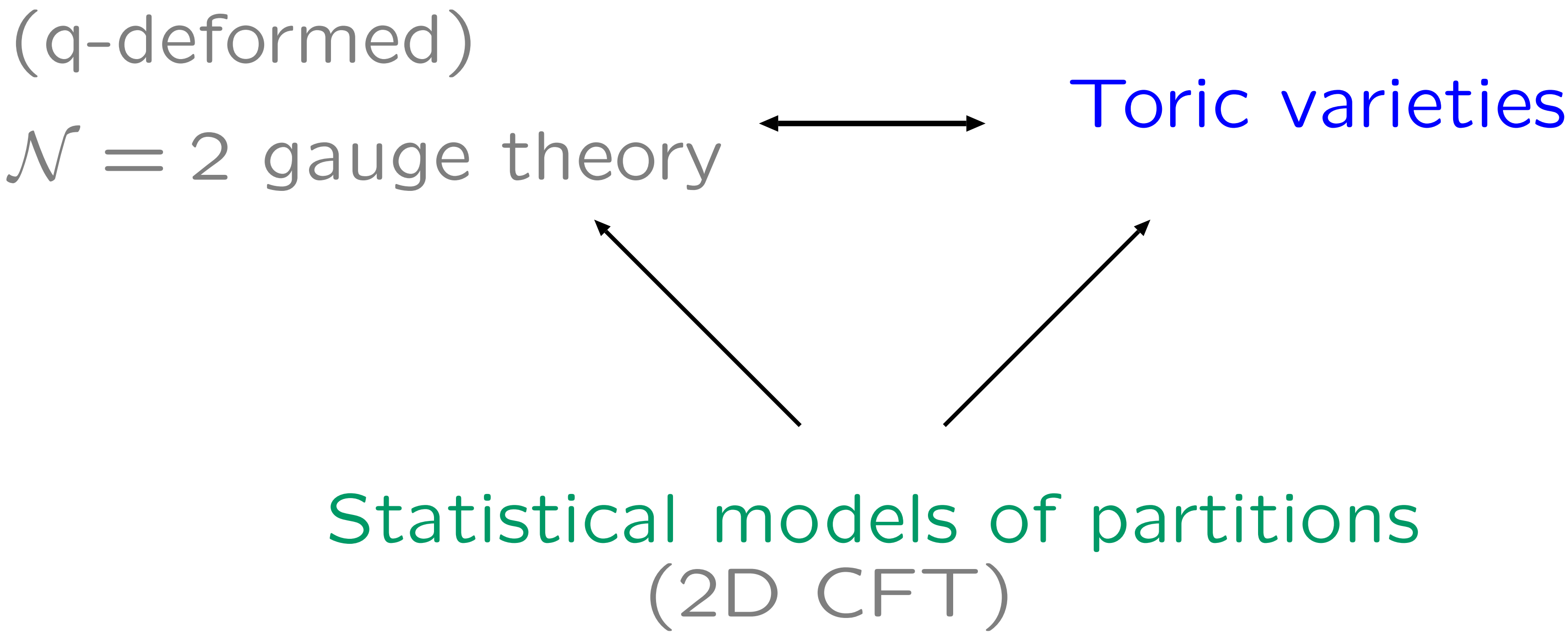

(q-deformed)
$\mathcal{N} = 2$ gauge theory
Toric varieties
Statistical models of partitions
(2D CFT)

$$Z^{inst}_{\text{Nek},5D\,adj} = Z^{inst}_{\text{SP}}(p),$$

$$\mathcal{F}^{pert}_{5D\,adj} = \Re\left(\lim_{\hbar\to 0}\hbar^2 \ln\left(\begin{array}{l}\left(\prod_{m=-\mu+1}^{\mu+1} q^{|\pi_{\text{GPP}}(m)|}\right)\\ \times(-zq^{-\mu})^{|2core|}(-q^{-\mu+1})^{|\pi_{\text{GPP}}(\mu)|}.\end{array}\right)\right)$$

$$+\text{const.} \qquad \text{for } \beta \gg 1,$$

$$\begin{aligned} q &= \exp(-\beta\hbar/2),\\ \mu &= 2m_{adj}/\hbar,\\ \ln z &= \exp(-8\pi^2/g^2_{YM}).\end{aligned}$$


(q-deformed)
$\mathcal{N} = 2$ gauge theory
Toric varieties
Statistical models of partitions
(2D CFT)

$\mathcal{P}_{adj}$ from π_{GPP}

$\mathcal{P}^c_{adj}$ emerges from π_{GPP} by the following map.

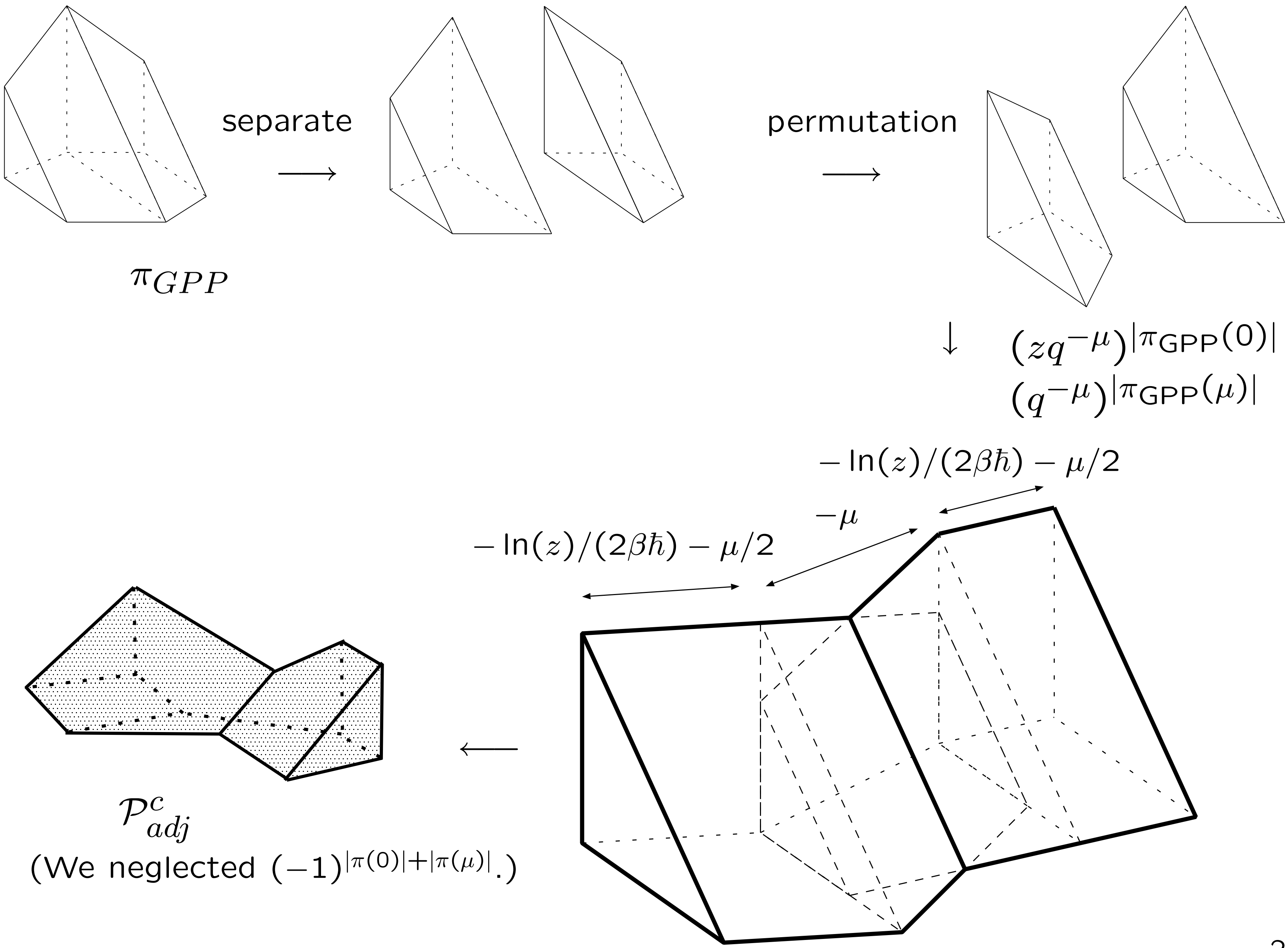
$$\Upsilon(n)_i = \begin{cases} \pi_{\text{GPP}}(0)_i & \text{if } \frac{\ln z}{2\beta\hbar} - \frac{\mu}{2} < n \leq -\mu \\ \max\{\pi_{\text{GPP}}(n)_i, \pi_{\text{GPP}}(\mu + n)_i\} & \text{if } -\mu < n \leq 0 \\ \pi_{\text{GPP}}(0)_i & \text{if } 0 < n \leq -\frac{\ln z}{2\beta\hbar} - \frac{\mu}{2}. \end{cases}$$

We can map bijectively from Υ to $m \in \mathcal{P}_{adj} \cap M$:

$$m = \begin{cases} ne_1^* + \frac{1}{N}(-\mu + j - i + 1)e_2^* + (\mu - n + i - 1)e_3^* \\ \qquad \text{for } \frac{\ln z}{2\beta\hbar} - \mu/2 < n \leq -\mu, \\ ne_1^* + \frac{1}{N}(n + j - i + 1)e_2^* + (-n + i - 1)e_3^* \\ \qquad \text{for } -\mu < n \leq 0, \\ ne_1^* + \frac{1}{N}(j - i + 1)e_2^* + (i - 1)e_3^* \\ \qquad \text{for } 0 < n \leq -\frac{\ln z}{2\beta\hbar} - \mu/2, \end{cases}$$

e_i^*: the basis of M

$\mathcal{P}_{adj}$ from π_{GPP}



Comment

Relation between the gauge theory, the statistical model and 2D CFT.

$$
\begin{aligned}
& Z_{\mathsf{Nek},5D\,adj\,U(1)} = Z_{\mathsf{SP}} \\
& = \sum_{\lambda,\nu} z^{\lambda} s_{\lambda/\nu}(q^{-i+\frac{\mu+1}{2}}) s_{\lambda^t/\nu^t}(q^{-i+\frac{\mu+1}{2}}) \\
& = \prod_{i=1}^{\infty} \left\{ (1-z^i)^{-1} \prod_{j,k=1}^{\mu} (1-z^i q^{-j-k-\mu+1}) \right\} \\
& = \mathsf{Tr} \left(z^{L_0} : \prod_{n=1}^{\mu} \mathsf{exp}(-i\varphi(q^{-n+\frac{\mu+1}{2}})) : \right).
\end{aligned}
$$

$s_{\lambda/\nu}(x^i)$: skew Schur function,

φ : 2D chiral free boson.

Summary

- We generalized the dualities to the case of the gauge theory with a massive adj. matter.

- This is a realization of gauge/gravity correspondece by means of statistical model of partitions.

Future direction

- Further genelarization to the case of quiver gauge theories.

- Relation between 2D CFT (WZW) and $SU(N)$ SYM with a massive adj. matter.

- Relation between integrable systems and SYM.

Appendix

4D $\mathcal{N}=2$ gauge theory

Multiplets in 4D $\mathcal{N}=2$ gauge theory

Vector multiplet	Hyper multiplet (matter)
A_μ	λ
$\psi^1 \quad \psi^2$	$X^1 \quad X^2$
ϕ	$\tilde{\lambda}$

The low energy effective action is determined by derivatives of a holomorphic function $\mathcal{F}$ called prepotential.

$$
\begin{aligned}
S &= \int \mathrm{d}^4 x \mathrm{d}^4 \theta \mathcal{F}(\Phi), \\
\mathcal{F} &= \mathcal{F}^{pert} + \mathcal{F}^{inst}.
\end{aligned}
$$

4D $\mathcal{N}=2$ gauge theory : Nekrasov formula

- Nekrasov formula for SU(2) with no matter

[Nekrasov and Okounkov '03]

$$Z_{\rm Nek} = Z^{pert}_{\rm Nek} \sum_{\lambda} \Lambda^{4|\lambda^{(1)}|+4|\lambda^{(2)}|} \prod_{(r,i)\neq(s,j)} \frac{a_{rs}/\hbar + \lambda_i^{(s)} - \lambda_j^{(r)} - i + j}{a_{rs}/\hbar + j - i},$$

$Z_{\rm Nek}$: Nekrasov's partition function

$\hbar$: graviphoton background $\simeq$ "string coupling"

λ : partition, $(\lambda = (\lambda_1, \lambda_2, \cdots),\ \lambda_i \in \mathbb{Z}_{\geq 0},\ \lambda_i \geq \lambda_{i+1})$,

$|\lambda| := \sum_{i=1}^{\infty} \lambda_i$

Λ : scale parameter

$a_{rs} := a_r - a_s$, a_s $(r = 1,2)$ is the vev. of the scalar $\sum_{r=1}^{2} a_r = 0$,

$|\lambda^1| + |\lambda^2|$: instanton number

$$\mathcal{F}^{pert} = \lim_{\hbar \to 0} \hbar^2 \ln Z^{pert}_{\rm Nek},$$
$$\mathcal{F}^{inst} = \lim_{\hbar \to 0} \hbar^2 (\ln Z_{\rm Nek} - \ln Z^{pert}_{\rm Nek}).$$

Nekrasov formula

- 5D ($\mathbb{R}^4 \times S^1$) generalized (q-deformed) Nekrasov formula

[Nekrasov and Okounkov '03]

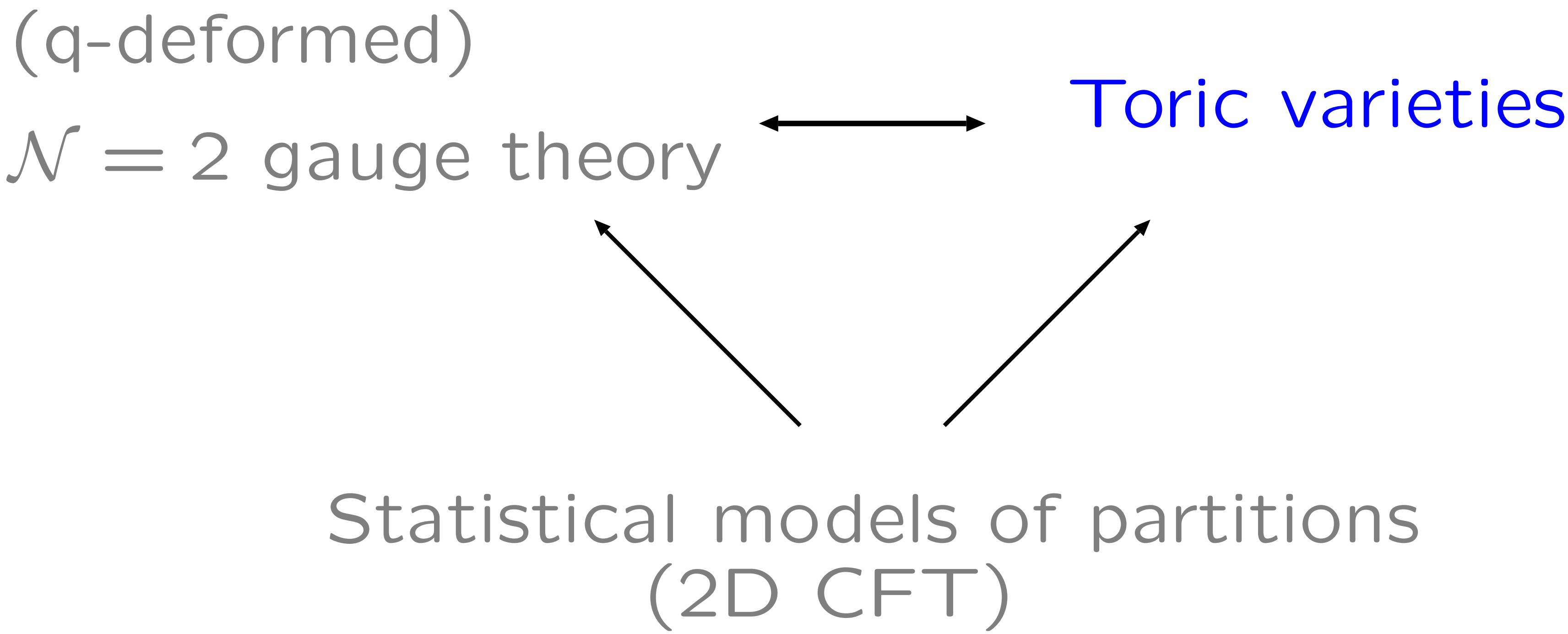
$$
\begin{aligned}
Z_{\text{Nek}\,5D} &= Z^{pert}_{\text{Nek}\,5D} \times \sum_{\lambda} (\beta\Lambda(q^{1/2} - q^{-1/2}))^{4|\lambda^{(1)}|+4|\lambda^{(2)}|} \\
&\quad \times \prod_{(r,i)\neq(s,j)} \frac{\left[2(a_{rs}/\hbar + \lambda_i^{(s)} - \lambda_j^{(r)} - i + j)\right]_{q^{1/2}}}{\left[2(a_{rs}/\hbar + j - i)\right]_{q^{1/2}}},
\end{aligned}
$$

β : circumference of S^1 in the 5th direction,

$[n]_{q^{1/2}} := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$, is called "$q$-integer",

$q := \exp(-\beta\hbar/2)$.

$$
\begin{aligned}
\mathcal{F}^{pert}_{5D} &= \lim_{\hbar\to 0} \hbar^2 \ln Z^{pert}_{\text{Nek},5D}, \\
\mathcal{F}^{inst}_{5D} &= \lim_{\hbar\to 0} \hbar^2 (\ln Z_{\text{Nek},5D} - \ln Z^{pert}_{\text{Nek},5D}).
\end{aligned}
$$


(q-deformed)
$\mathcal{N} = 2$ gauge theory
Toric varieties
Statistical models of partitions
(2D CFT)

Toric variety

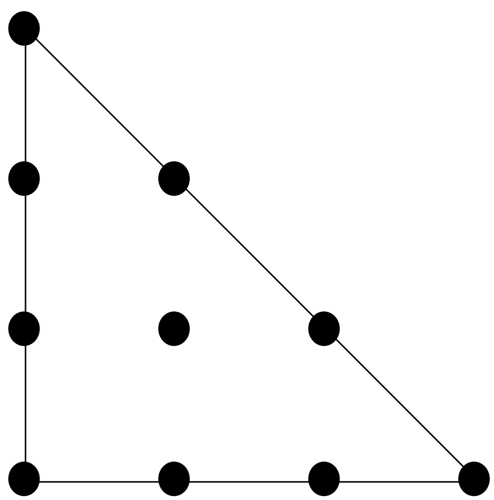
$\mathbb{C}^*$: algebraic torus

Toric variety is obtained by adding points and so on to $(\mathbb{C}^*)^n$.

Example:
$\mathbb{C} = \mathbb{C}^* \cup \{0\}$,
$\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$.

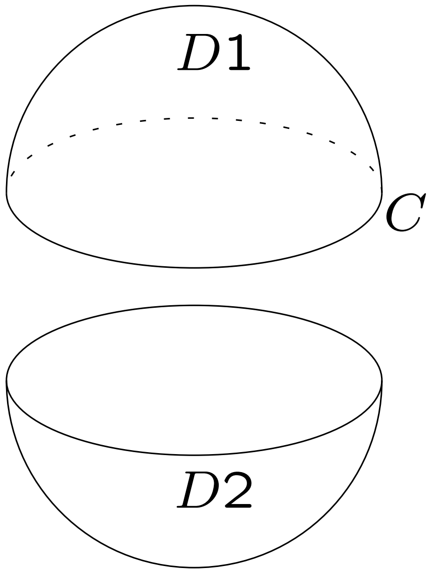
All toric varieties and sections of holomolphic line bundles on it are described by polyhedrons on a lattice M.

Example:



$\mathbb{CP}^2$

Geometric quantization (Bohr-Sommerfeld quantization)



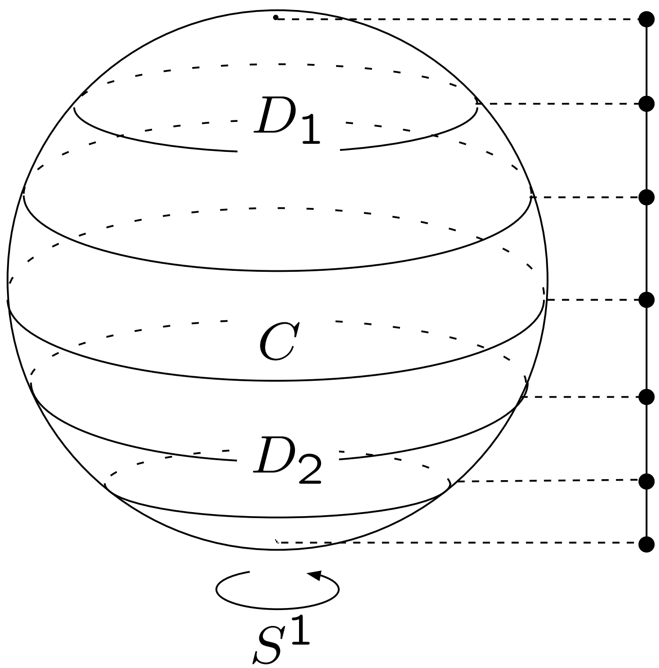
Bohr-Sommerfeld quantization rule

$$\mathbb{Z} \ni \frac{1}{h} \oint_C p \, \mathrm{d}q \;\; = \;\; \frac{1}{h} \int_{D_{1,2}} \pm\omega \, ,$$

$$\omega := \mathrm{d}p \wedge \mathrm{d}q,$$

$$\Rightarrow T = \frac{1}{h} \int_{D_1 \cup D_2} \omega \;\; \in \;\; \mathbb{Z}$$

Toric variety: Geometric quantization (Bohr-Sommerfeld quantization)



A point = Gravitational quantum foam

ω: Kähler two form
C: S^1 orbit

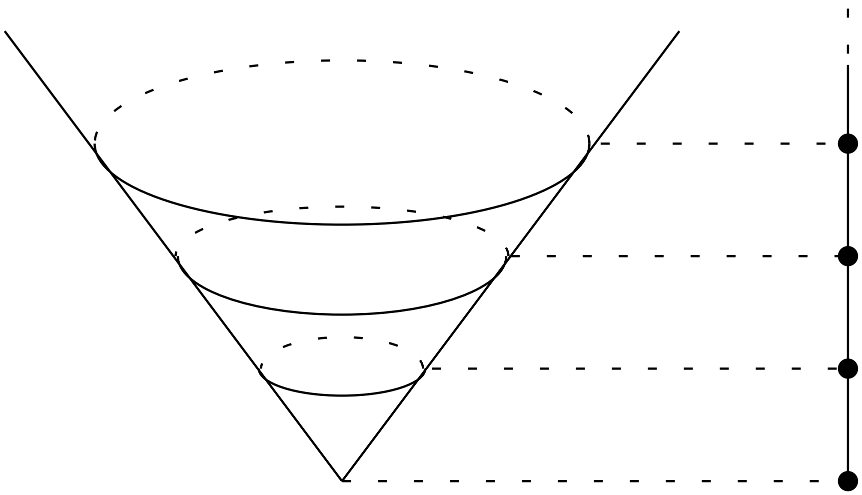
$$\mathbb{Z} \ni \frac{1}{g_{st}} \oint_C p\, dq \;=\; \frac{1}{g_{st}} \int_{D_{1,2}} \pm\omega$$

$$\Rightarrow T = \frac{1}{g_{st}} \int_{D_1 \cup D_2} \omega \;\in\; \mathbb{Z}$$

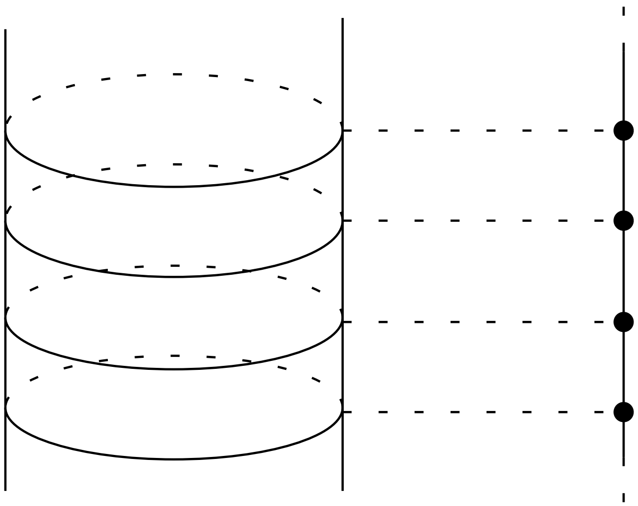
Kähler parameter is quantized
in the unit of string coupling $g_{st}\alpha'$ ($\alpha' = 1$, $g_{st} = \beta\hbar$).

Cardinality is the dimension of the Hilbert space.

Toric variety: Generalization to noncompact varieties

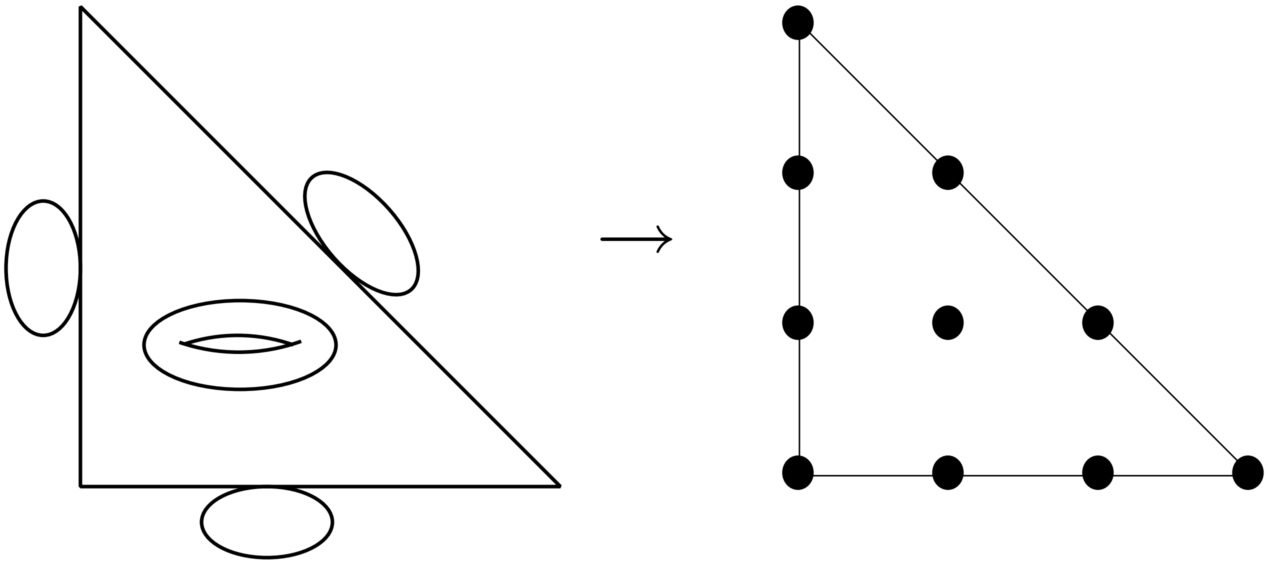


$\mathbb{C}$ is characterized by half line.

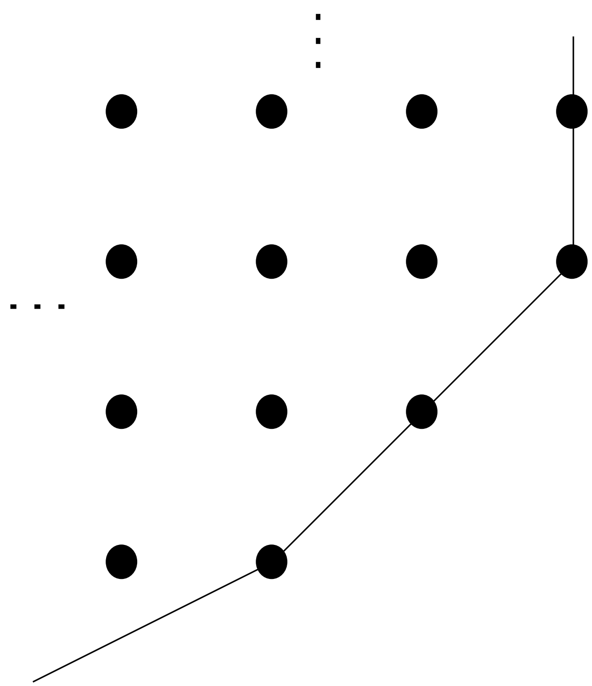


$\mathbb{C}^*$ is characterized by line.

Toric variety: Two dimensional example

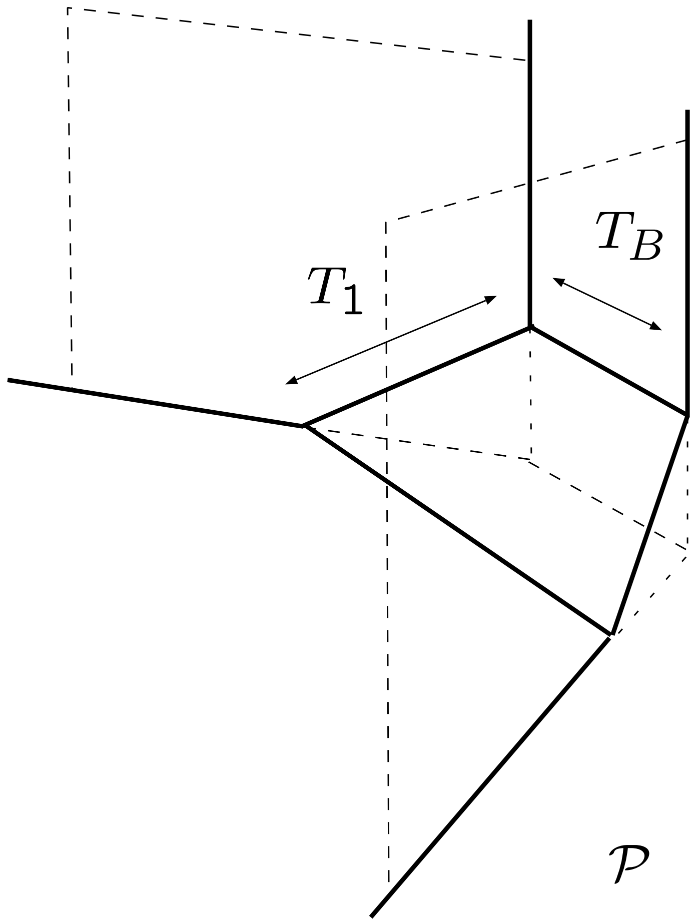


$\mathbb{CP}^2$



ALE space (A_1) ($\to \mathbb{C}^2/\mathbb{Z}_2$)

Polyhedron

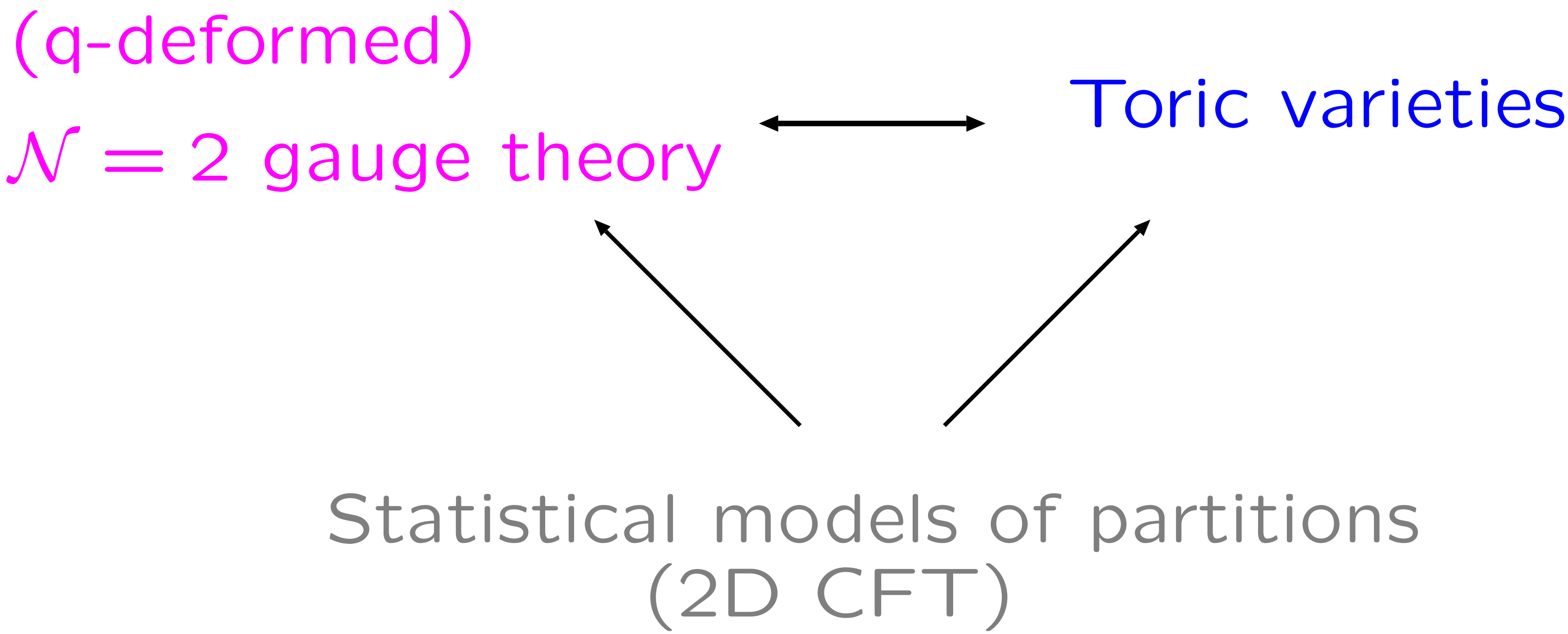


Polyhedron $\mathcal{P}$ for X.

$$\begin{array}{ccc} X & \leftarrow & \text{ALE space } (A_1) \\ \downarrow & & \\ \mathbb{CP}^1 & & \text{non cpt. C.Y. 3-fold} \end{array}$$

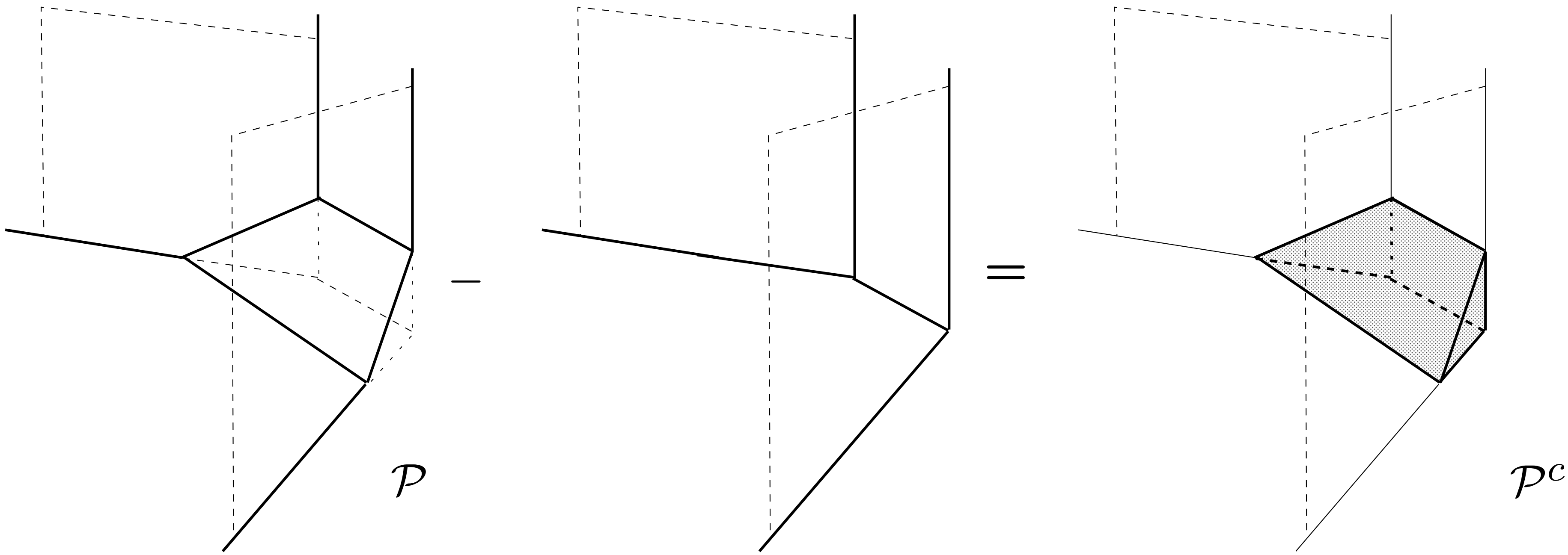
T_B, T_1 : quantized Kähler parameters

X is a resolution of the singularities at the origin of the metric cone of $Y^{2,2}$.


(q-deformed)
$\mathcal{N} = 2$ gauge theory
Toric varieties
Statistical models of partitions
(2D CFT)

Prepotential from Polyhedron

We regularize the cardinality. [Maeda, Nakatsu, Y.N. and Tamakoshi '05]



$$\mathcal{F}_{5D}^{pert} = \lim_{\hbar \to 0} \hbar^2 \mathrm{Card}(\mathcal{P}^c \cap M) + \text{const.}, \qquad \text{for } \beta \gg 1,$$

$$\begin{aligned} T_1 &= 2a_2, \\ T_B &= \frac{-4 \ln(\beta\Lambda)}{\beta\hbar}. \end{aligned}$$

$\beta\hbar T_B$ and $\beta\hbar T_1$ are fixed.

adding adjoint matter

Gauge theory with a massive adjoint matter

∘ Nekrasov's partition function with a massive adj. matter

[Nekrasov and Okounkov '03]

$$Z_{\mathrm{Nek}\,adj} = Z^{pert}_{\mathrm{Nek}\,adj} \sum_{\lambda} z^{4|\lambda^{(1)}|+4|\lambda^{(2)}|}$$
$$\times \prod_{(r,i)\neq(s,j)} \frac{(a_{rs}/\hbar + \lambda_i^{(s)} - \lambda_j^{(r)} - i + j)\cdot(\frac{m_{adj}+a_{rs}}{\hbar} + +j - i)}{(\frac{a_{rs}}{\hbar} + j - i)\cdot(\frac{m_{adj}+a_{rs}}{\hbar} + \lambda_i^{(s)} - \lambda_j^{(r)} - i + j)},$$

$z = \exp(-8\pi^2/g^2_{YM})$, m_{adj}: mass of the adj matter

∘ 5D generalization (q-deformation)

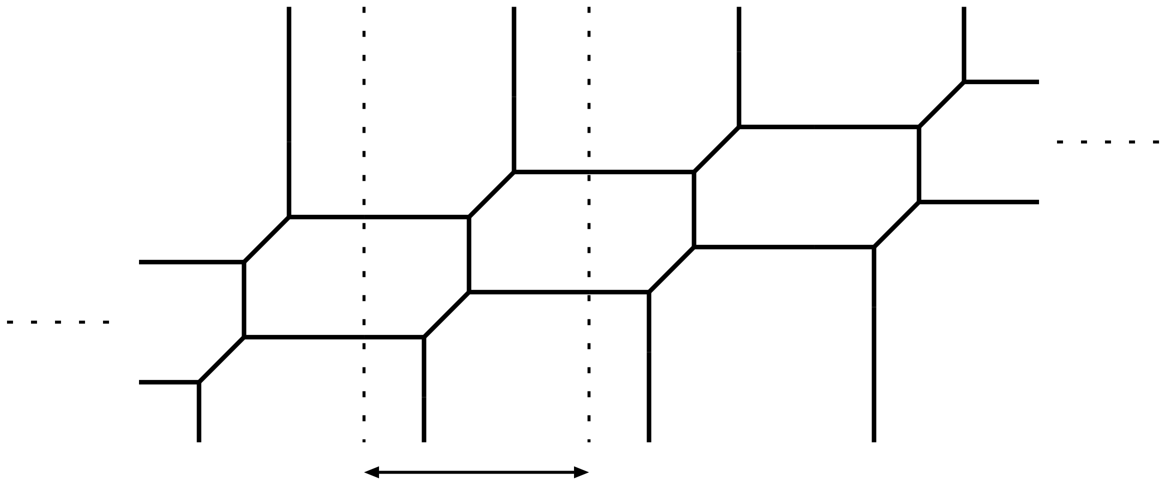
$$Z_{\mathrm{Nek}\,5D\,adj} = Z^{pert}_{\mathrm{Nek}\,5D\,adj} \sum_{\lambda} z^{4|\lambda^{(1)}|+4|\lambda^{(2)}|}$$
$$\times \prod_{(r,i)\neq(s,j)} \frac{\left[2(\frac{a_{rs}}{\hbar} + \lambda_i^{(s)} - \lambda_j^{(r)} - i + j)\right]_{q^{1/2}} \cdot \left[2(\frac{\mu}{2} + \frac{a_{rs}}{\hbar} + j - i)\right]_{q^{1/2}}}{\left[2(\frac{a_{rs}}{\hbar} + j - i)\right]_{q^{1/2}} \cdot \left[2(\frac{\mu}{2} + \frac{a_{rs}}{\hbar} + \lambda_i^{(s)} - \lambda_j^{(r)} - i + j)\right]_{q^{1/2}}}$$

$\mu = \frac{2m_{adj}}{\hbar}$.

Polyhedron

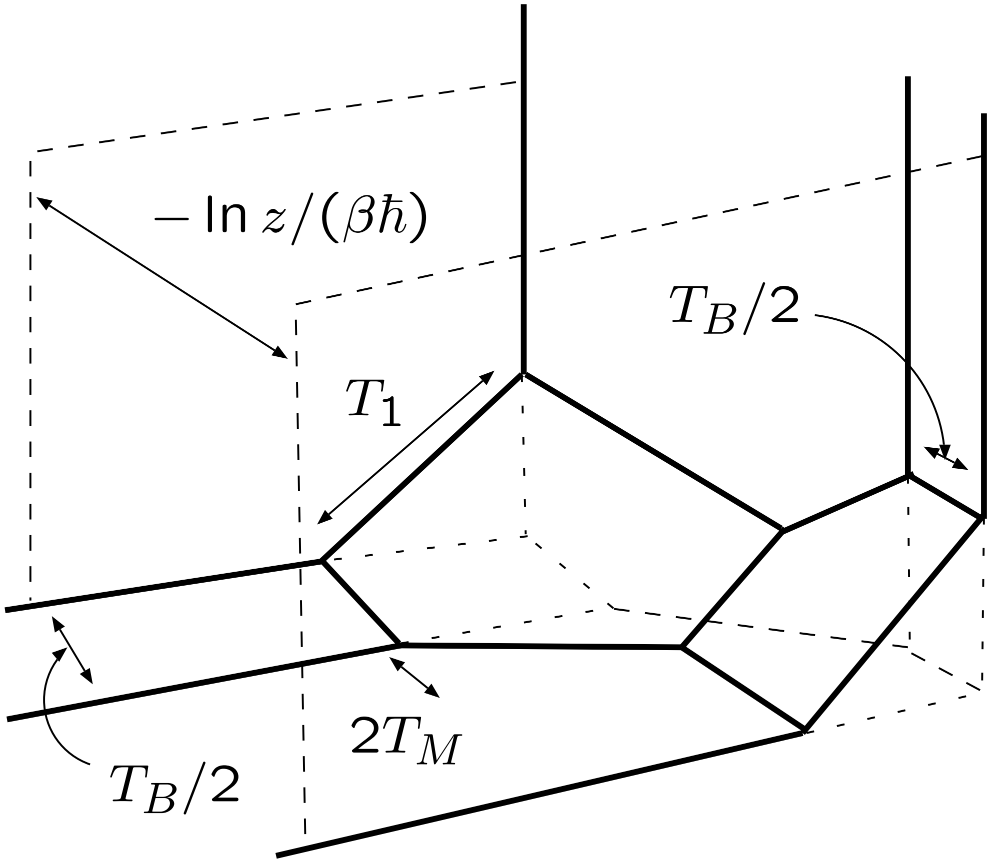
We want to consider $\begin{array}{c} X_{adj} \\ \downarrow \\ \text{“}T^2\text{”} \end{array}$ $\longleftarrow$ ALE space (A_1)

non cpt. C.Y. 3-fold



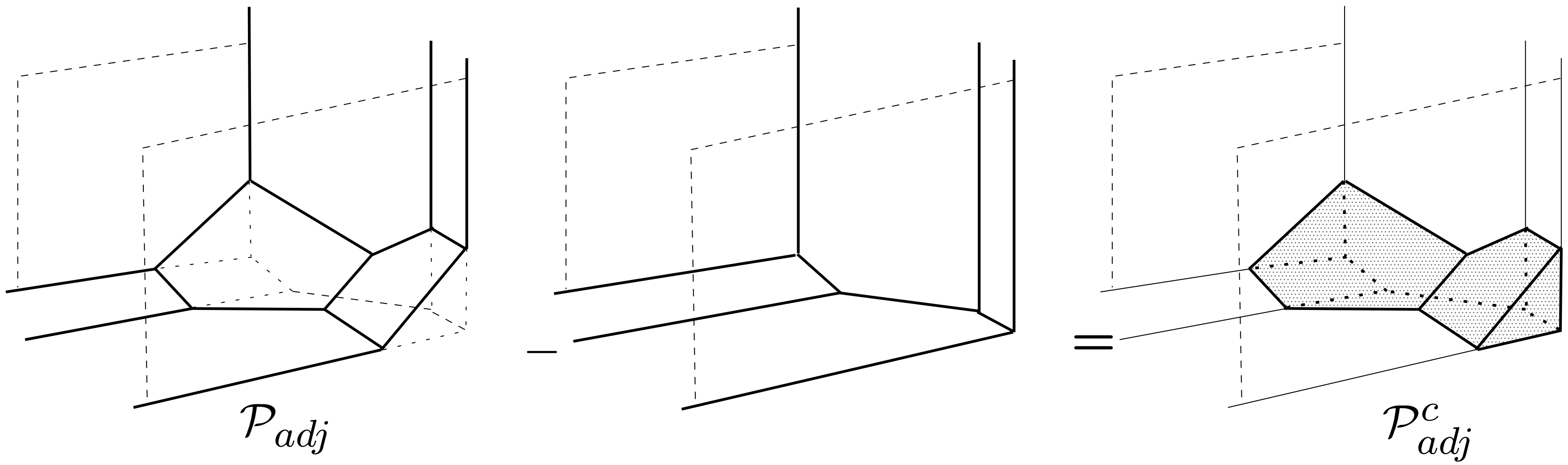
We consider a certain polyhedron surrounded by infinite planes and regard it periodic.

Polyhedron



We regard $\mathcal{P}_{adj} \cap M$ as the Hilbert space of X_{adj}.

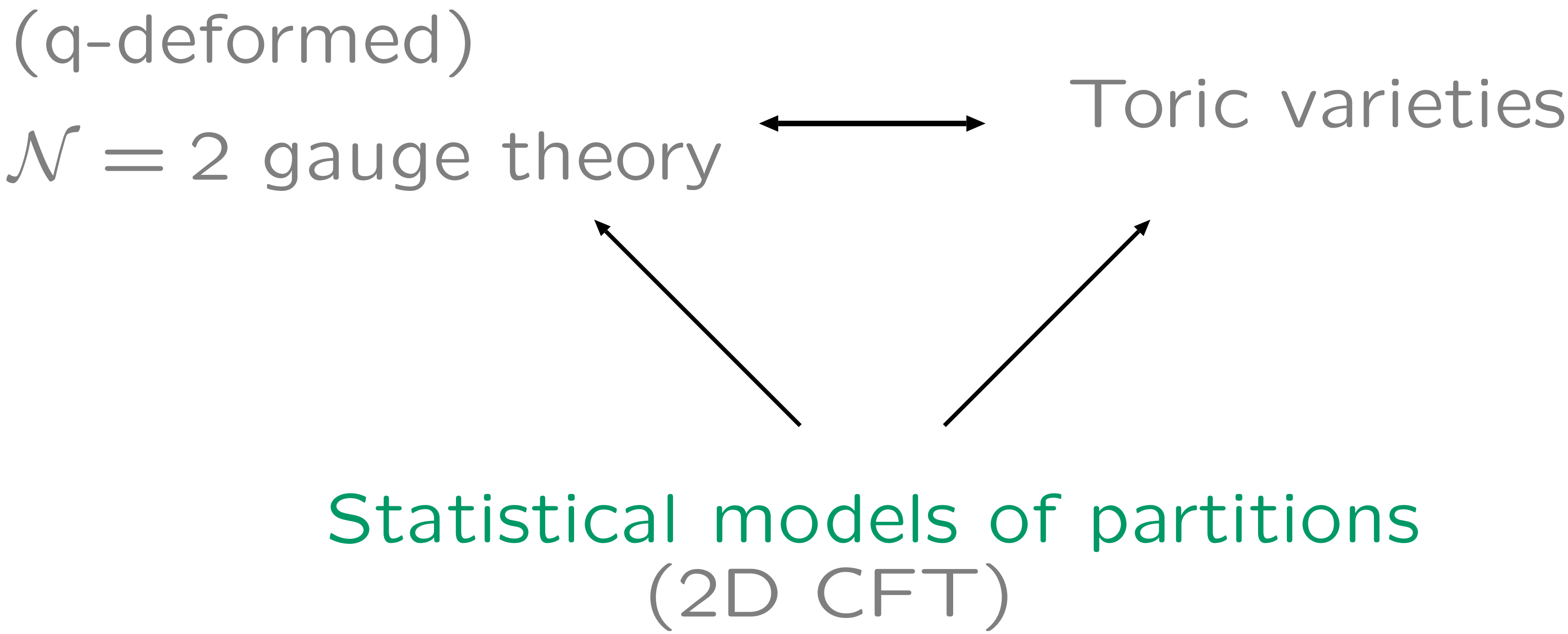
Prepotential from Polyhedron



$$\mathcal{F}^{pert}_{5D\,adj} = \lim_{\hbar \to 0} \hbar^2 \mathrm{Card}(\mathcal{P}^c_{adj} \cap M) + \text{const.}, \qquad \text{for } \beta \gg 1.$$

$$\begin{aligned} T_M &= m_{adj}/\hbar, \\ T_B &= -\ln z/(\beta\hbar) - 2T_M, \\ \ln z &= \exp(-8\pi^2/g^2_{YM}). \end{aligned}$$

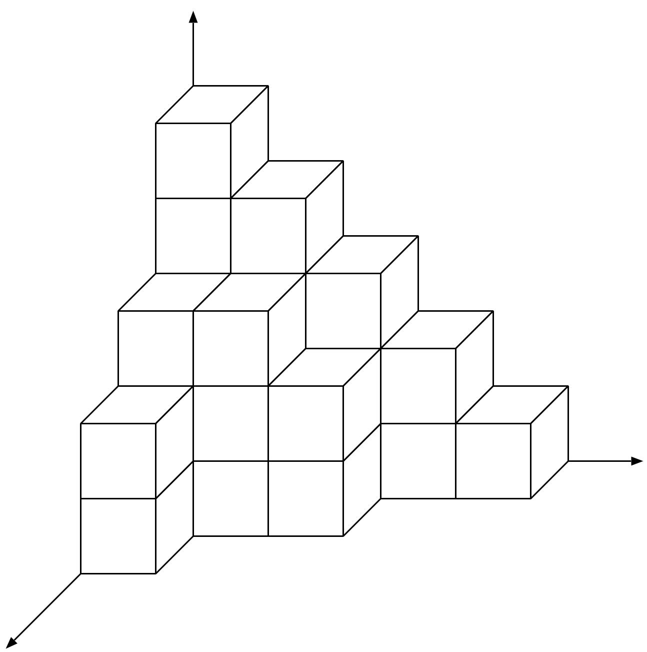
$\beta\hbar T_B$, $\beta\hbar T_1$ $\beta\hbar T_M$ are fixed.


(q-deformed)
$\mathcal{N} = 2$ gauge theory
Toric varieties
Statistical models of partitions
(2D CFT)

Statistical model of partitions : Plane partition

- Plane partition $\pi = \{\pi_{i\,j}\}_{i,j=1}^{\infty}$, $\pi_{i\,j} \in \mathbb{Z}_{\geq 0}$,

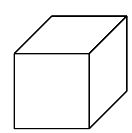
$$
\begin{array}{ccccccc}
\pi_{1\,1} & \geq & \pi_{1\,2} & \geq & \pi_{1\,3} & \geq & \cdots \\
\text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \\
\pi_{2\,1} & \geq & \pi_{2\,2} & \geq & \pi_{2\,3} & \geq & \cdots \\
\text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \\
\pi_{3\,1} & \geq & \pi_{3\,2} & \geq & \pi_{3\,3} & \geq & \cdots \\
\text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$



$$
\pi = \begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & & \\ 2 & & & & \end{pmatrix}
$$

plane partition

(3D Young diagram)



$:U(1)$ instanton

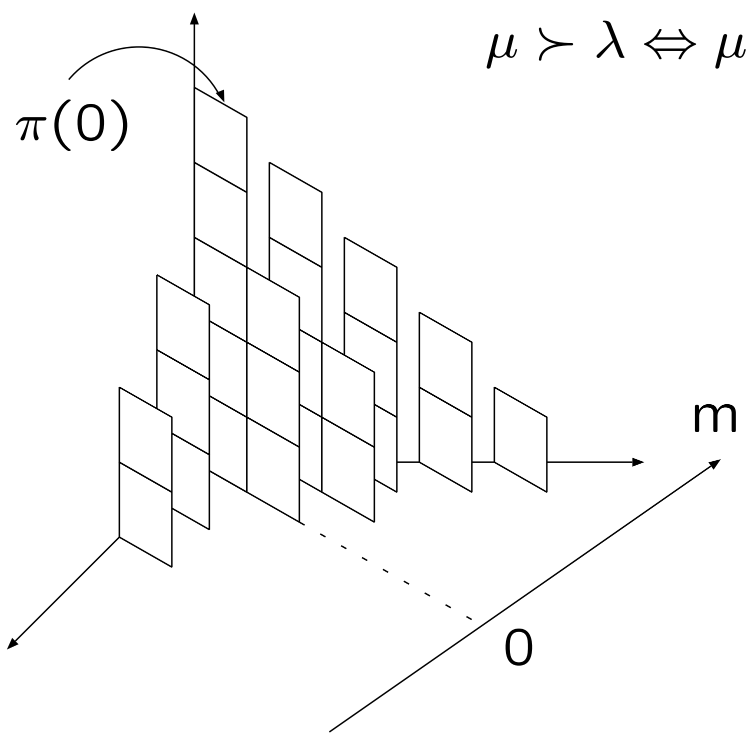
◦ Diagonal slice of plane partition

Plane partition can be seen as a sequence of partitions:

$$\pi(m) = \begin{cases} (\pi_{1+m\,1}, \pi_{2+m\,2}, \cdots) & \text{for } m \geq 0, \\ (\pi_{1-m\,1}, \pi_{2-m\,2}, \cdots) & \text{for } m < 0. \end{cases}$$

$$\cdots \prec \pi(-2) \prec \pi(-1) \prec \pi(0) \succ \pi(1) \succ \pi(2) \succ \cdots ,$$

$$\mu \succ \lambda \Leftrightarrow \mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots .$$



$$\begin{aligned} \pi(4) &= (1), \\ \pi(3) &= (2), \\ \pi(2) &= (3), \\ \pi(1) &= (4,2), \\ \pi(0) &= (5,3), \\ \pi(-1) &= (3), \\ \pi(-2) &= (2). \end{aligned}$$

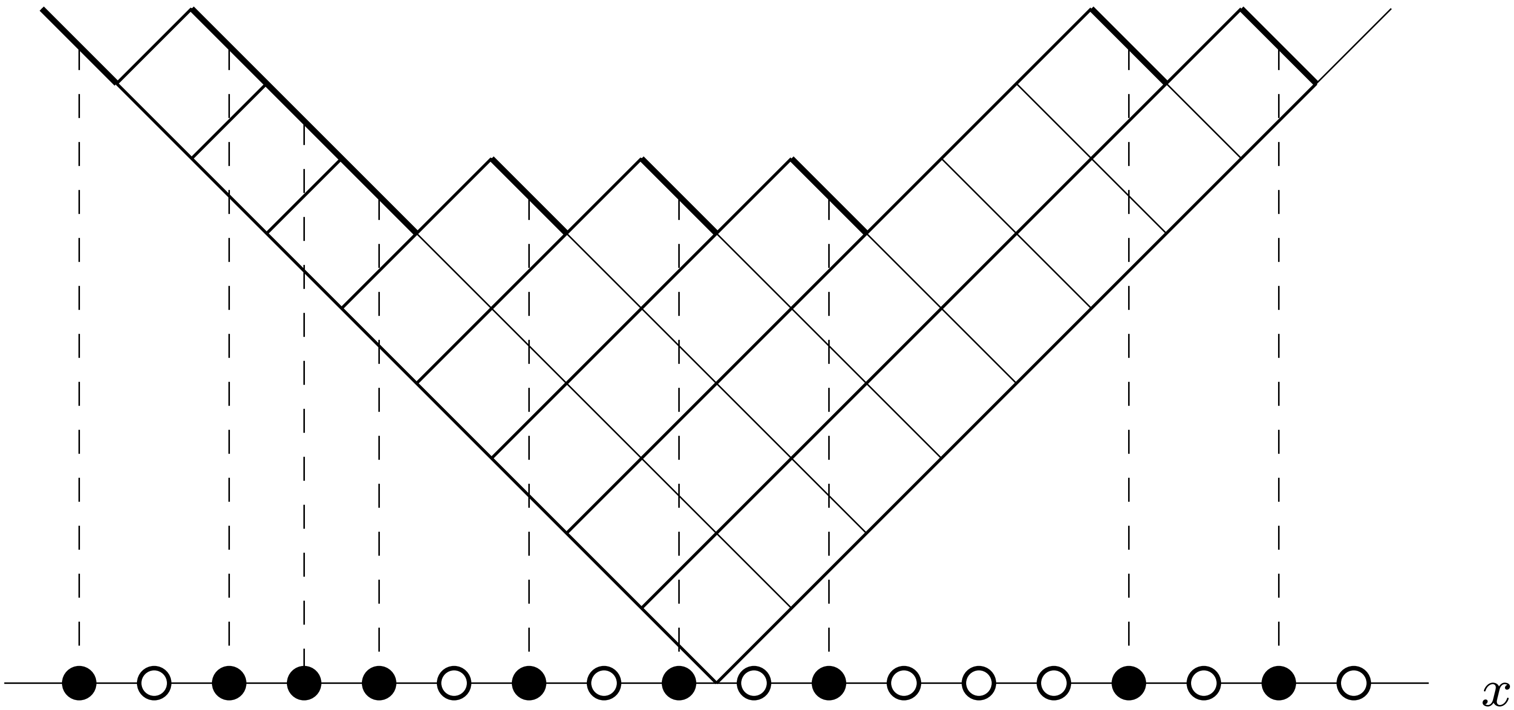
∘ Partition function of random plane partition model

$$
\begin{aligned}
Z_{\mathsf{RPP}}(q,Q) &:= \sum_{\pi} q^{|\pi|} Q^{|\pi(0)|} \\
&= \sum_{\lambda} Q^{|\lambda|} \left(\sum_{\substack{m=-\infty \\ \pi(0)=\lambda}}^{\infty} q^{|\pi(m)|} \right) \\
&= \sum_{\lambda} Q^{|\lambda|} (s_{\lambda}(q^{-\rho}))^2
\end{aligned}
$$

$|\pi|$: # of boxes of π,
Q $:= (\beta\Lambda)^2$,
s_{λ} : Schur function
$q^{-\rho}$ $:= (q^{1/2}, q^{3/2}, \cdots)$.

For a partition ν, we can define a Maya diagram.

$$x_i(\nu) := \nu_i - i + \frac{1}{2}.$$



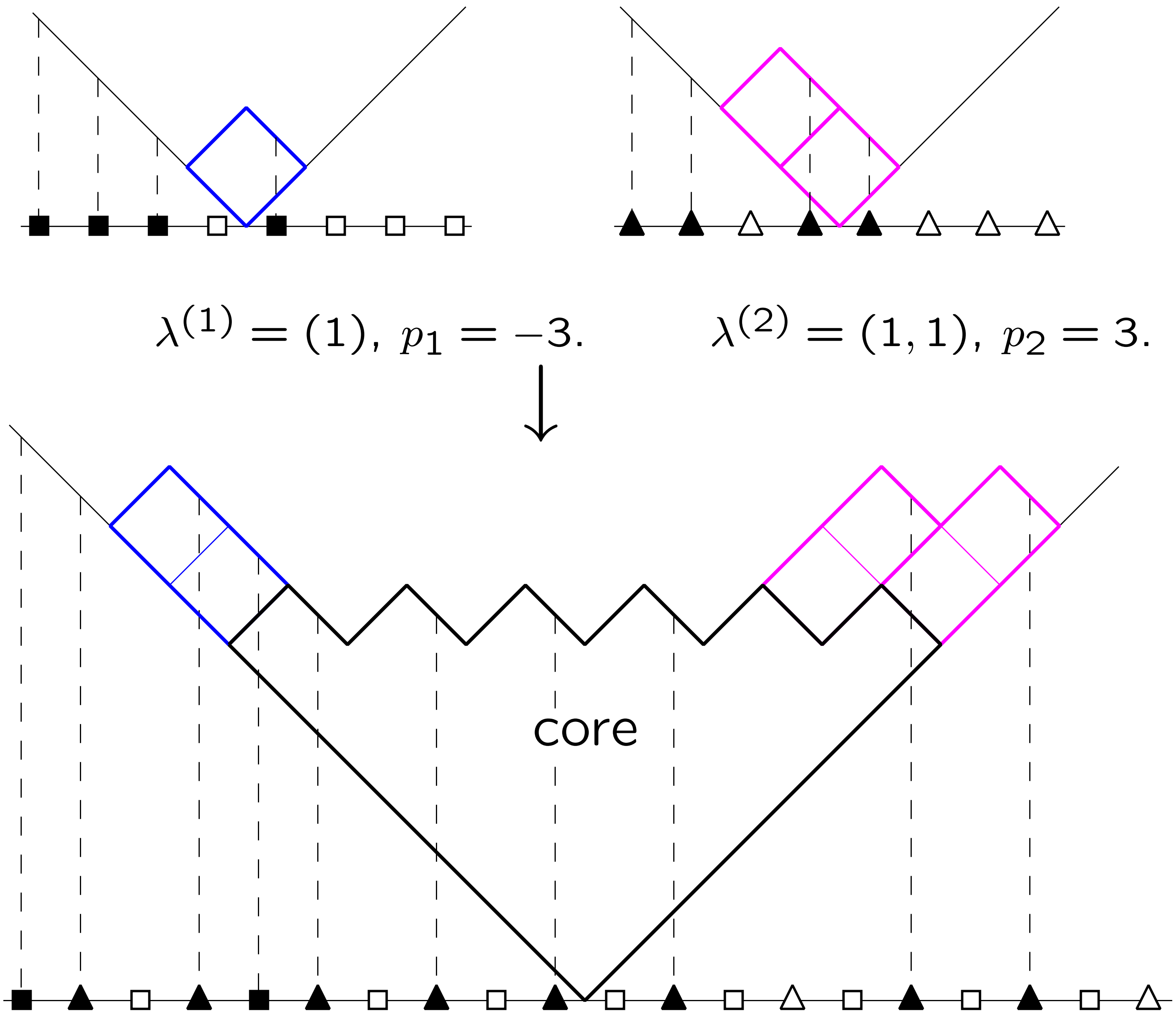
Maya diagram

$(\nu = (8, 7, 4, 3, 2, 1, 1, 1))$

Two partition $\lambda^{(r)}$, $r = 1, 2$ can be embedded to a single partition ν s.t.

$$\left\{x_i(\nu(\lambda^{(r)}, p_r)); i \geq 1\right\} = \bigcup_{r=1}^{2} \left\{2(x_{i_r}(\lambda^{(r)}) + \tilde{p}_r); i_r \geq 1\right\},$$

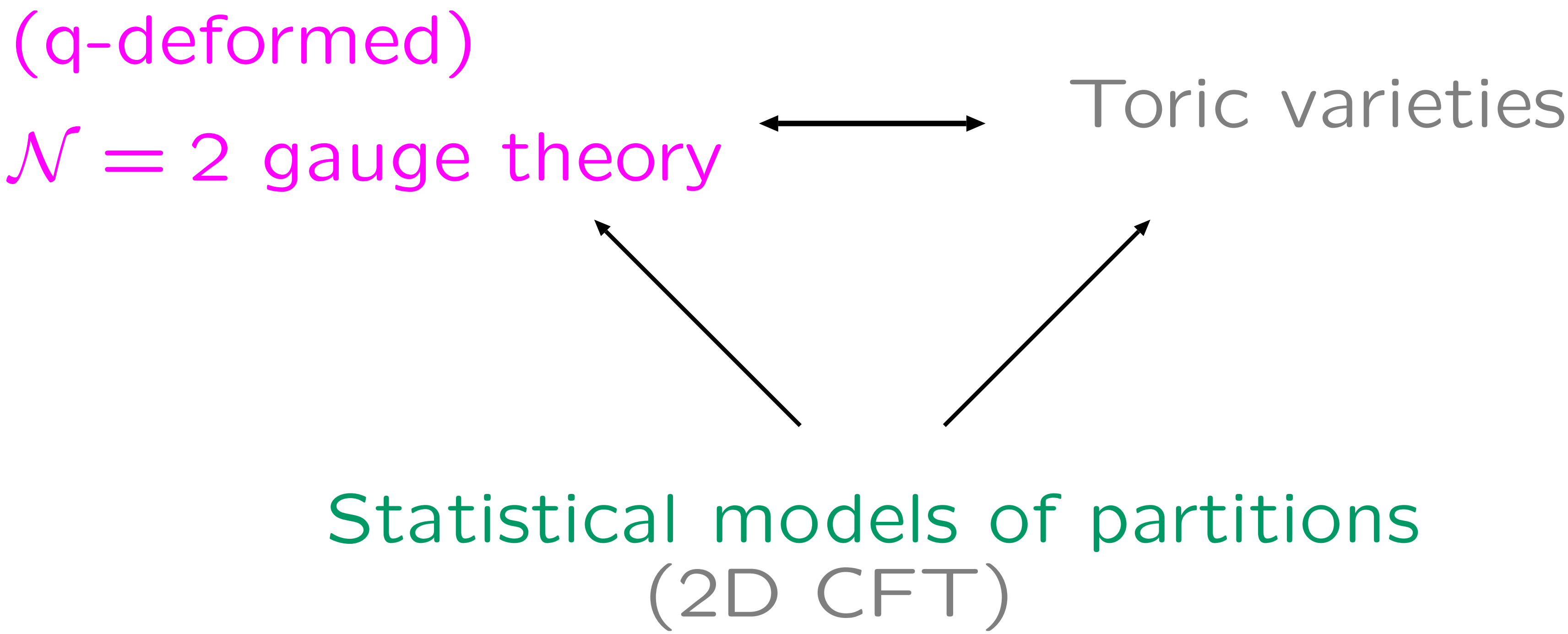
$\lambda^{(r)}$: r-th partition, p_r : charge for r-th parititon,
$\tilde{p}_r$ $:= p_r + \xi_r$, ξ_r $:= \frac{1}{2}(r - \frac{3}{2})$.


$\lambda^{(1)} = (1)$, $p_1 = -3$.
$\lambda^{(2)} = (1,1)$, $p_2 = 3$.
core

Because the mapping is bijective, we obtain :

$$\sum_{i=1}^{\infty} e^{tx_i(\nu)} = \sum_{r=1}^{2} \sum_{i=1}^{\infty} e^{2t(x_{i_r}(\lambda^{(r)})+\tilde{p}_r)},$$

$$\Rightarrow \begin{aligned} 0 &= \sum_{r=1}^{2} p_r, \\ |\nu(\lambda^{(r)}, p_r)| &= 2\sum_{r=1}^{2} |\lambda^{(r)}| + \sum_{r=1}^{2} p_r^2 + \sum_{r=1}^{2} r p_r, \\ &\vdots \end{aligned}$$


(q-deformed)
$\mathcal{N} = 2$ gauge theory
Toric varieties
Statistical models of partitions
(2D CFT)

Prepotential from Random plane partition model

$Z_{\text{RPP}}(q, Q)$ can be factorized to two parts:

$$Z_{\text{RPP}}(q,Q) = \sum_{p} \sum_{\lambda^{(1)},\lambda^{(2)}} Q^{|\text{core}|+2|\lambda^{(1)}|+2|\lambda^{(2)}|} \left(\sum_{\substack{m=-\infty \\ \pi(0)=\lambda(\text{core},\lambda^{(1)},\lambda^{(2)})}}^{\infty} q^{|\pi|} \right)$$

$$= \sum_{p} Z^{pert}_{\text{RPP}}(q,Q,p) \cdot Z^{inst}_{\text{RPP}}(q,Q,p).$$

$$Z^{pert}_{\text{RPP}}(q,Q,p) := Q^{|\text{core}|} \left(\sum_{\substack{m=-\infty \\ \pi(0)=\text{core}}}^{\infty} q^{|\pi|} \right),$$

$p = -p_1 = p_2$.

The prepotential emerges from Z_{RPP}.

[Maeda, Nakatsu, Takasaki and Tamakoshi '04]

$$\mathcal{F}^{pert}_{5D} = \lim_{\hbar \to 0} \hbar^2 \ln Z^{pert}_{\text{RPP}}(q,Q,p) + \text{const.},$$

$$Z^{inst}_{\text{Nek},5D} = Z^{inst}_{\text{RPP}}(q,Q,p),$$

$\tilde{p}_2 = a_2/\hbar$.

Prepotential from Random plane partition model

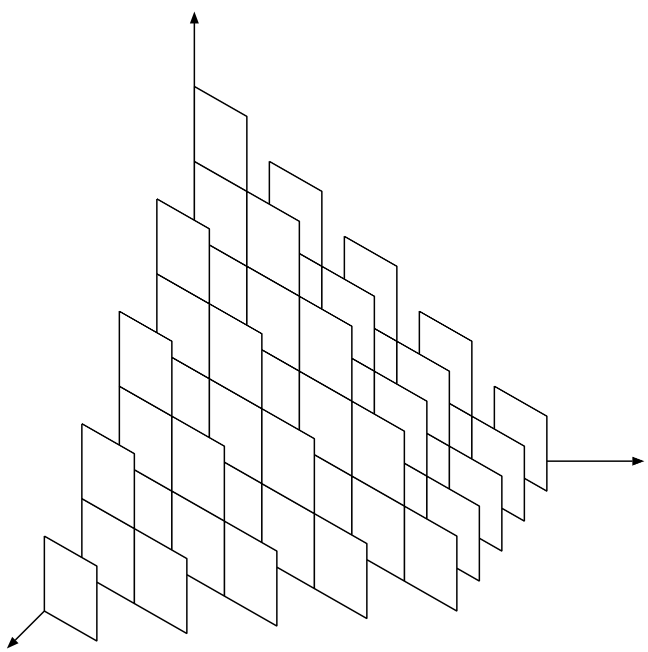
In particular, $\exists \pi_{\text{GPP}}$ s.t.

$\pi_{\text{GPP}}(0) = \text{core}$ and $|\pi_{\text{GPP}}| \leq |\pi|$ for all $\pi|_{\pi(0)=\text{core}}$.

π_{GPP} dominates Z^{pert}_{RPP} at $q \to 0$ ($\beta \to \infty$).

[Maeda, Nakatsu, Y.N. and Tamakoshi '05]

$$\mathcal{F}^{pert}_{5D} = \lim_{\hbar \to 0} \hbar^2 \ln q^{|\pi_{\text{GPP}}|} Q^{|\pi_{\text{GPP}}(0)|} + \text{const.}, \qquad \text{for } \beta \gg 1$$



π_{GPP} for the case of $p = 5$.

(q-deformed)
$\mathcal{N} = 2$ gauge theory

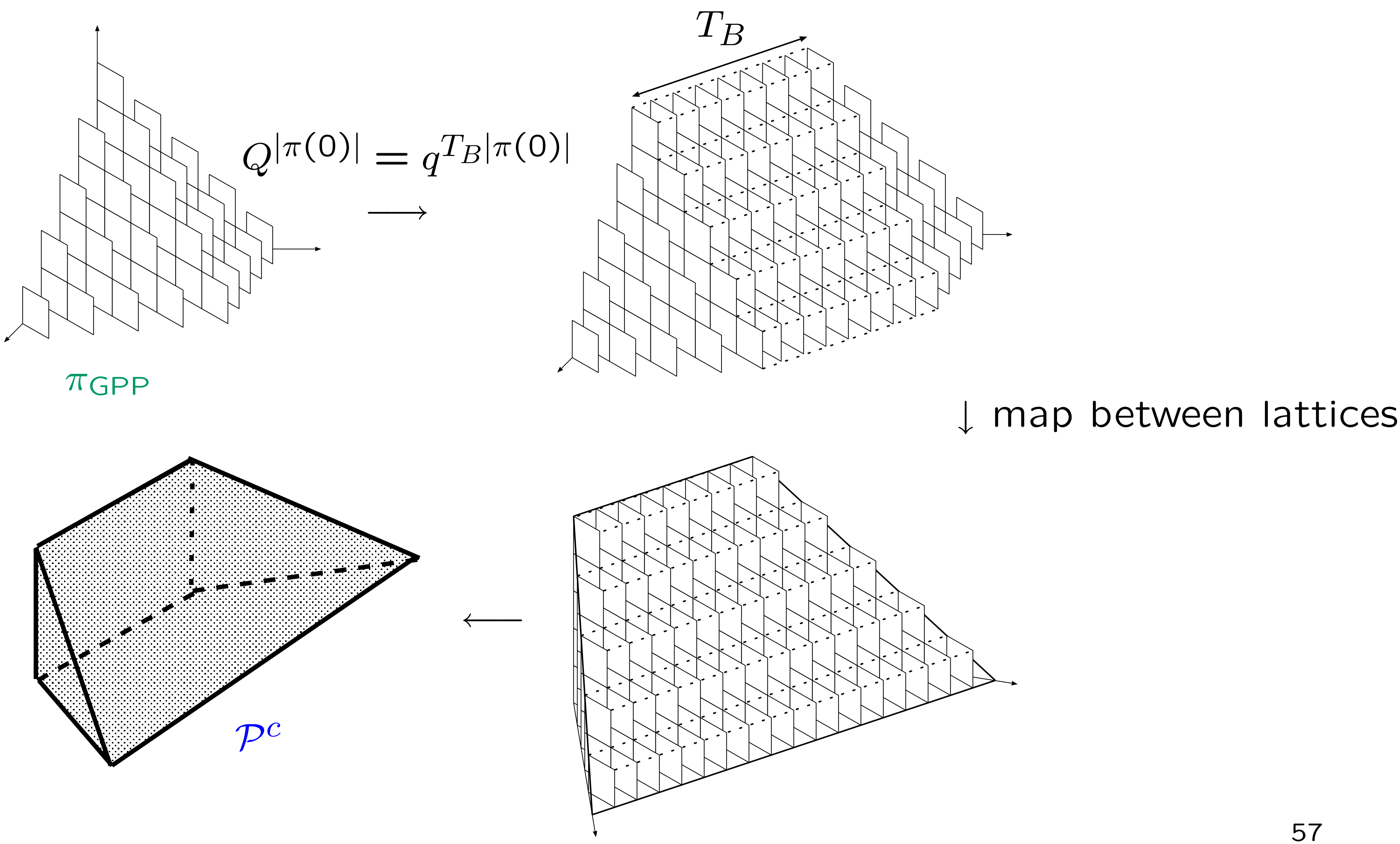
Toric varieties

Statistical models of partitions
(2D CFT)

Polyhedron from π_{GPP}

$\mathcal{P}$ emerges from π_{GPP} as follows.

[Maeda, Nakatsu, Y.N. and Tamakoshi '04]



adding adjoint matter

Statistical model of partitions

- π a sequence of partitions *s.t.*

$$\pi(-\mu) \prec \cdots \prec \pi(-2) \prec \pi(-1) \prec \pi(0),$$
$$\pi(0)^t \succ \pi(1)^t \succ \pi(2)^t \succ \cdots \succ \pi(\mu)^t,$$
$$\pi(\mu) = \pi(-\mu).$$

$$Z_{\mathsf{SP}} := \sum_{\pi} \left(\prod_{m=-\mu+1}^{\mu+1} q^{|\pi(m)|} \right) (-zq^{-\mu})^{|\pi(0)|} (-q^{-\mu+1})^{|\pi(\mu)|}.$$
$$= \sum_{\text{2core}} Z_{\mathsf{SP}}^{pert}(p) \cdot Z_{\mathsf{SP}}^{inst}(p),$$

$$Z_{\mathsf{SP}}^{pert}(p) := \sum_{\substack{\pi \\ \pi(0)=\text{2core}}} \left(\prod_{m=-\mu+1}^{\mu+1} q^{|\pi(m)|} \right) \times (-zq^{-\mu})^{|\pi(0)|} (-q^{-\mu+1})^{|\pi(\mu)|}.$$

$$P_{GP}(p) := \{\pi | \pi(0) = \mathrm{core}(p)\}\,.$$

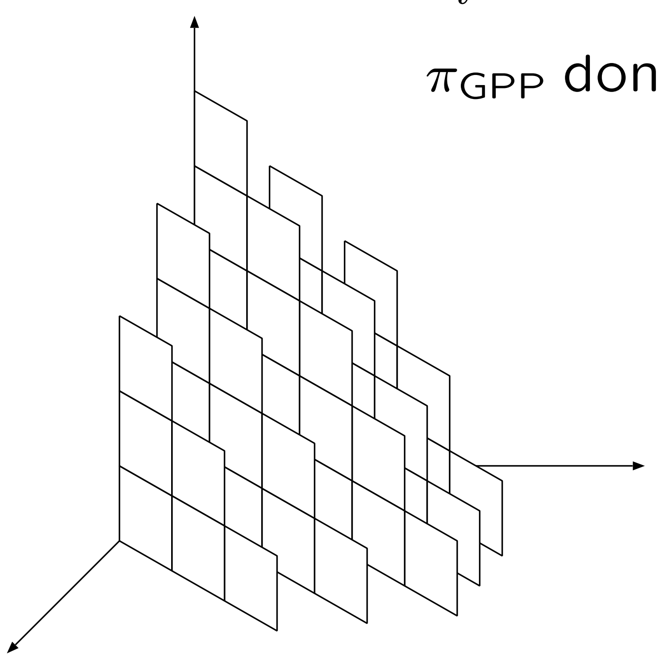
$$^{\exists}\pi_{\mathsf{GPP}} \in P_{GP}(p),\ \ s.t.\ |\pi_{\mathsf{GPP}}| \leq |\pi|,\ \ ^{\forall}\pi \in P_{GP}(p).$$

The explicit form is

$$\pi_{\mathsf{GPP}\,i}(n) \;=\; \begin{cases} \max\{\mathrm{core}(p)_i - n,\ \lambda_i^{\mu}\} & \text{for } (n \geq 0) \\ \max\{\mathrm{core}(p)_{i+n},\ \lambda_i^{\mu}\} & \text{for } (n < 0), \end{cases}$$

$$\lambda_i^{\mu} :\;=\; \max\{\mathrm{core}(p)_{i+\mu}, \mathrm{core}_i - \mu\}.$$

π_{GPP} donimates $Z_{\mathsf{SP}}^{pert}(p)$ at $q \to 0$ $(\beta \to \infty)$.



π_{GPP} in the case of $\mu = 2$

$$
\begin{aligned}
Z^{inst}_{\text{Nek},5D\,adj} &= Z^{inst}_{\text{SP}}(p), \\
\mathcal{F}^{pert}_{5D\,adj} &= \Re\left(\lim_{\hbar\to 0}\hbar^2 \ln\left(\begin{array}{l}\left(\prod_{m=-\mu+1}^{\mu+1} q^{|\pi_{\text{GPP}}(m)|}\right) \\ \times(-zq^{-\mu})^{|2core|}(-q^{-\mu+1})^{|\pi_{\text{GPP}}(\mu)|}.\end{array}\right)\right) \\
& \quad +\text{const.} \qquad\qquad \text{for } \beta \gg 1,
\end{aligned}
$$

$$
\begin{aligned}
q &= \exp(-\beta\hbar/2), \\
\mu &= 2m_{adj}/\hbar, \\
\ln z &= \exp(-8\pi^2/g^2_{YM}).
\end{aligned}
$$

$\mathcal{P}_{adj}$ from π_{GPP}

$\mathcal{P}^c_{adj}$ emerges from π_{GPP} by the following map.

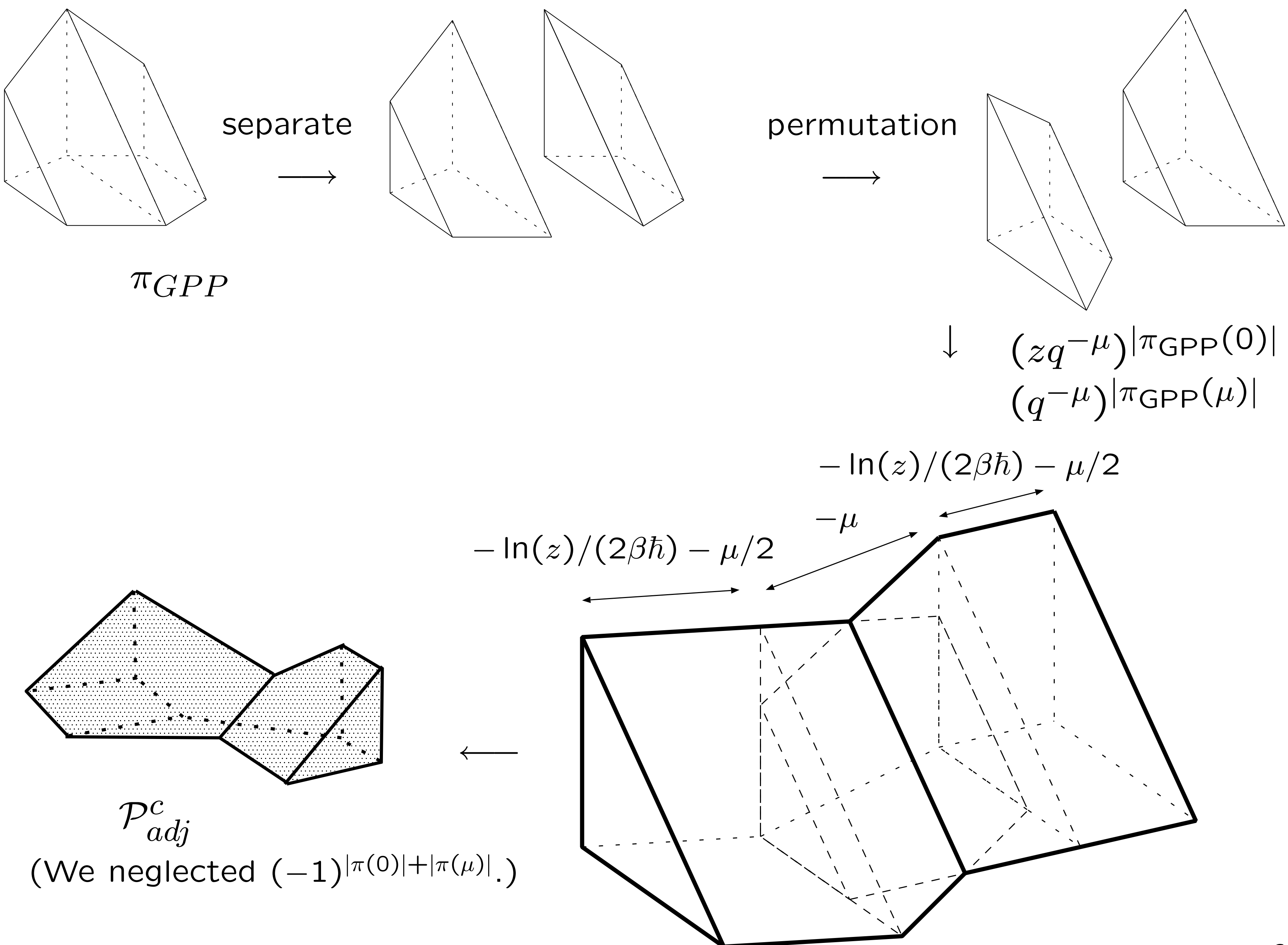
$$\Upsilon(n)_i = \begin{cases} \pi_{\mathrm{GPP}}(0)_i & \text{if } \frac{\ln z}{2\beta\hbar} - \frac{\mu}{2} < n \leq -\mu \\ \max\{\pi_{\mathrm{GPP}}(n)_i, \, \pi_{\mathrm{GPP}}(\mu + n)_i\} & \text{if } -\mu < n \leq 0 \\ \pi_{\mathrm{GPP}}(0)_i & \text{if } 0 < n \leq -\frac{\ln z}{2\beta\hbar} - \frac{\mu}{2}. \end{cases}$$

We can map bijectively from Υ to $m \in \mathcal{P}_{adj} \cap M$:

$$m = \begin{cases} n e_1^* + \frac{1}{N}(-\mu + j - i + 1) e_2^* + (\mu - n + i - 1) e_3^* \\ \qquad \text{for } \frac{\ln z}{2\beta\hbar} - \mu/2 < n \leq -\mu, \\ n e_1^* + \frac{1}{N}(n + j - i + 1) e_2^* + (-n + i - 1) e_3^* \\ \qquad \text{for } -\mu < n \leq 0, \\ n e_1^* + \frac{1}{N}(j - i + 1) e_2^* + (i - 1) e_3^* \\ \qquad \text{for } 0 < n \leq -\frac{\ln z}{2\beta\hbar} - \mu/2, \end{cases}$$

e_i^*: the basis of M

$\mathcal{P}_{adj}$ from π_{GPP}



Comment

Relation between the gauge theory, the statistical model and 2D CFT.

$$
\begin{aligned}
& Z_{\mathrm{Nek},5D\,adj\,U(1)} = Z_{\mathrm{SP}} \\
& = \sum_{\lambda,\nu} z^{\lambda} s_{\lambda/\nu}(q^{-i+\frac{\mu+1}{2}}) s_{\lambda^t/\nu^t}(q^{-i+\frac{\mu+1}{2}}) \\
& = \prod_{i=1}^{\infty} \left\{ (1-z^i)^{-1} \prod_{j,k=1}^{\mu} (1 - z^i q^{-j-k-\mu+1}) \right\} \\
& = \mathrm{Tr} \left(z^{L_0} : \prod_{n=1}^{\mu} \exp(-i\varphi(q^{-n+\frac{\mu+1}{2}})) : \right).
\end{aligned}
$$

$s_{\lambda/\nu}(x^i)$: skew Schur function,

φ : 2D chiral free boson.