On the concavity of a-functions

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Zonotopes and four-dimensional superconformal field theories

Motivation

- To understand the universality classes and RG flows of four dimensional QFTs a la

  Theorem. [Zamoldchikov 1986] *There exists a real-valued function* $c : \mathcal{M}_{2dCFT} \longrightarrow \mathbb{R}$ *such that the RG flow is a gradient line of* c*-function:* $\beta^i(g) = -G(g)^{ij}\dfrac{\partial c(g)}{\partial g^j}$*. The critical value of* c *is the Virasoro central charge of the corresponding CFT.*

- Toward a proof of AdS/CFT correspondence or gauge/gravity duality

4d Superconformal Field Theories and $U(1)_R$

- Global symmetry contains $SU(2,2|1) \supset SO(4,2) \times U(1)_R$
- Scaling dimension of chiral operators are protected from quantum corrections: $\Delta(\mathcal{O}) = \frac{3}{2}R(\mathcal{O})$.
- Conjecture : a-function defined by $a = \frac{3}{32}\left(3\,\mathrm{tr}\,R^3 - \mathrm{tr}\,R\right)$ decreases along RG flow: $a_{UV} > a_{IR}$.
- $U(1)_R$ is thus extremely useful provided that it can be correctly identified. In general, however, abelian part of non-R global flavor symmetry G can mix with $U(1)_R$.

a-maximization

Theorem. [Intriligator-Wecht 2003]
Exact $U(1)_R$ charges maximize a:
Among all possible combination of abelian currents

$$R_\phi = R_0 + \sum_{i=1}^{n} \phi^i F_i,$$

the correct $U(1)_R$ current is given by the ϕ which attains the maximum of the "trial" a-function

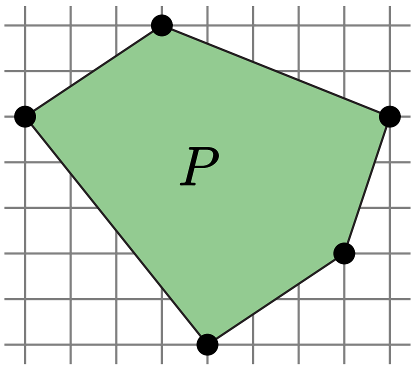
$$a(\phi) = \frac{3}{32}\big(3\,\mathrm{tr}\,R_\phi^3 - \mathrm{tr}\,R_\phi\big).$$

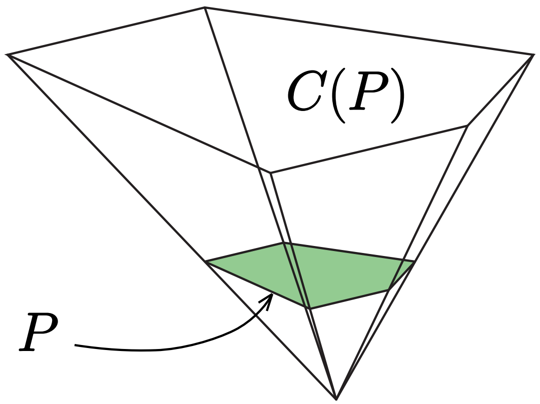
a-functions from toric diagrams

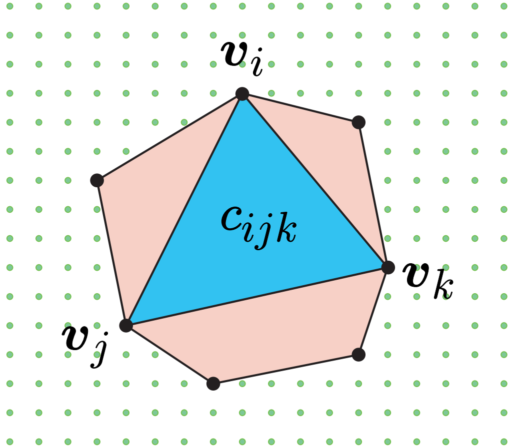
For a toric diagram P with vertices $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$, the a function of the corresponding quiver gauge theory is given by

$$a(\phi) = \frac{9}{32}\frac{N^2}{2}\sum_{i,j,k=1}^{n} c_{ijk}\phi^i\phi^j\phi^k, \qquad c_{ijk} = |\det(\boldsymbol{v}_i, \boldsymbol{v}_j, \boldsymbol{v}_k)|,$$

where $\phi^1 + \cdots + \phi^n = 2, \phi^i > 0.$







Hanany-Iqbal, Benvenuti-Franco-Hanany-Martelli-Sparks, Butti-Zaffaroni, Franco-Hanany-Kennaway-Vegh-Wecht, Benvenuti-Kruczenski, ...

Benvenuti-Zayas-Tachikawa, Lee-Rey

Basic Questions

- Does a-maximization always have a solution?
- Is it unique? No saddle points?
- Do non-extremal points in toric diagrams play their role in a-maximization?
- How does the change of toric diagrams influence the maxima of trial a-functions? Does a-function decrease whenever a toric diagram shrinks?
- Don't want to build conjectures upon other conjectures....
  Can we answer these questions without assuming AdS/CFT correspondence?

Mathematical Setup

- Input data : a toric diagram P with vertices $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$.
- The trial a-function $\hat{F}_P : \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$\hat{F}_P(\phi) = \sum_{1 \le i < j < k \le n} |\det(\boldsymbol{v}_i, \boldsymbol{v}_j, \boldsymbol{v}_k)| \, \phi^i \phi^j \phi^k.$$

- Physical range of R-charges
$\Gamma_n := \left\{ (\phi^1, \cdots, \phi^n) \in \mathbb{R}^n \ : \ \phi^i \ge 0, \ \sum_{i=1}^n \phi^i = r \right\} \subset \mathbb{R}^n.$
- $\boxed{\text{Extremize the function } F_P : \Gamma_n \longrightarrow \mathbb{R}.}$
- Modulus := normalized maximum value of a-function

$$\mathfrak{M}(P) := \left(\frac{3}{r}\right)^3 \max_{\phi \in \Gamma_n} \hat{F}_P(\phi).$$

Main Theorems

- **Existence and Uniqueness**
  Let P be a toric diagram with vertices $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_n$. Then the function $F_P : \Gamma_n \to \mathbb{R}$ has a unique critical point ϕ_* in the relative interior of Γ_n and ϕ_* is also the unique global maximum of F_P.

- **Universal Upper Bound**
  The critical point ϕ_* satisfies the universal bound $0 < \phi_*^i \leq r/3$ for all i. The equality $\phi_*^i = r/3$ holds for some i if and only if $n = 3$.

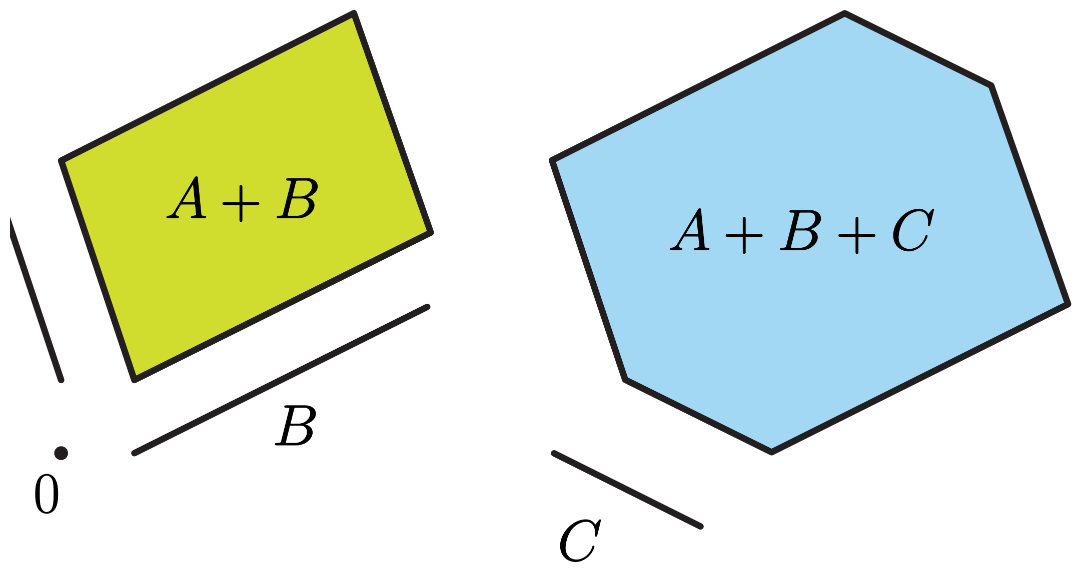
- **Monotonicity**
  The maximum value $\mathfrak{M}(P)$ depends on P only through its convex hull. $\mathfrak{M}(P)$ is monotone in the sense that if $P \subset P'$ up to integral affine transformations $G := GL(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$, then $\mathfrak{M}(P) \leq \mathfrak{M}(P')$. The equality holds if and only if $P = P'$ up to G-action.

Polytopes and Minkowski sums

- Convex hull of $S \subset \mathbb{R}^d$
  $\operatorname{conv}(S) := \{\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in \mathbb{R}^d \ : \ \boldsymbol{x}, \boldsymbol{y} \in S, \ 0 \leq \lambda \leq 1\}.$

- Minkowski sum of $A, B \subset \mathbb{R}^d$
  $A + B := \{\boldsymbol{x} + \boldsymbol{y} \ : \ \boldsymbol{x} \in A, \ \boldsymbol{y} \in B\}.$

- Dilatation $rA := \{r\boldsymbol{x} \ : \ \boldsymbol{x} \in A\}.$



Zonotopes

For a vector configuration $X = \{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n\} \subset \mathbb{R}^d$, the zonotope is given by

$$\mathcal{Z}(X) = \{\boldsymbol{x} \in \mathbb{R}^d \ : \ \boldsymbol{x} = \lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_n \boldsymbol{x}_n, \ 0 \le \lambda_i \le 1\}$$

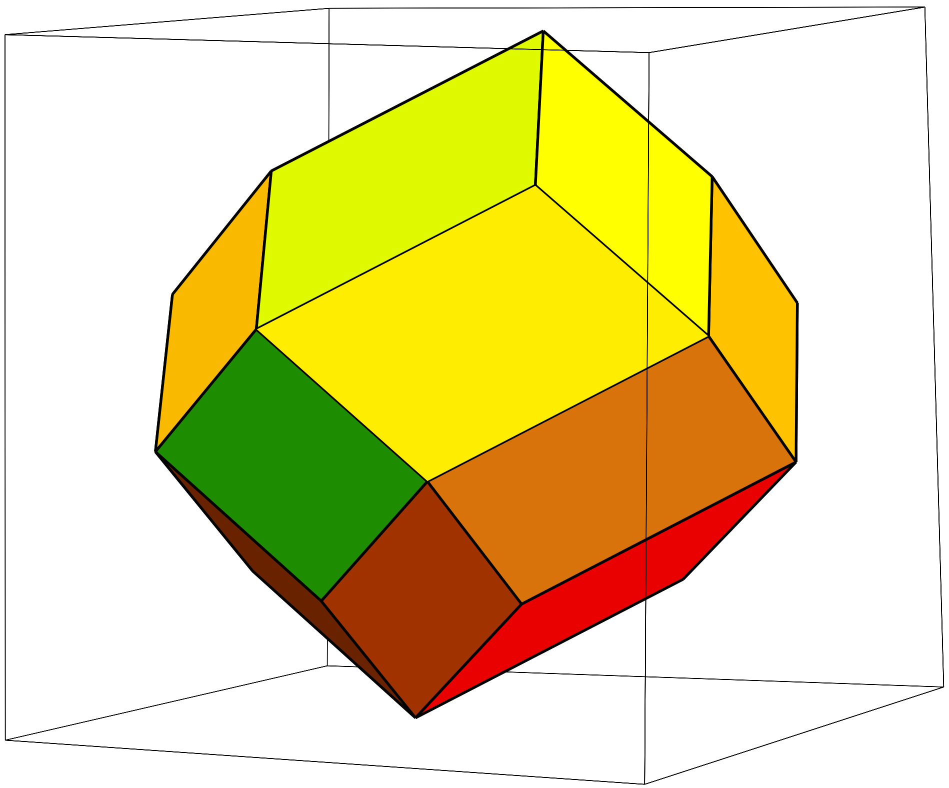
Theorem. [Shephard, McMullen]

$$\mathrm{vol}_d(\mathcal{Z}(X)) = \sum_{1 \le i_1 < \cdots < i_d \le n} |\det(\boldsymbol{x}_{i_1}, \cdots, \boldsymbol{x}_{i_d})|.$$

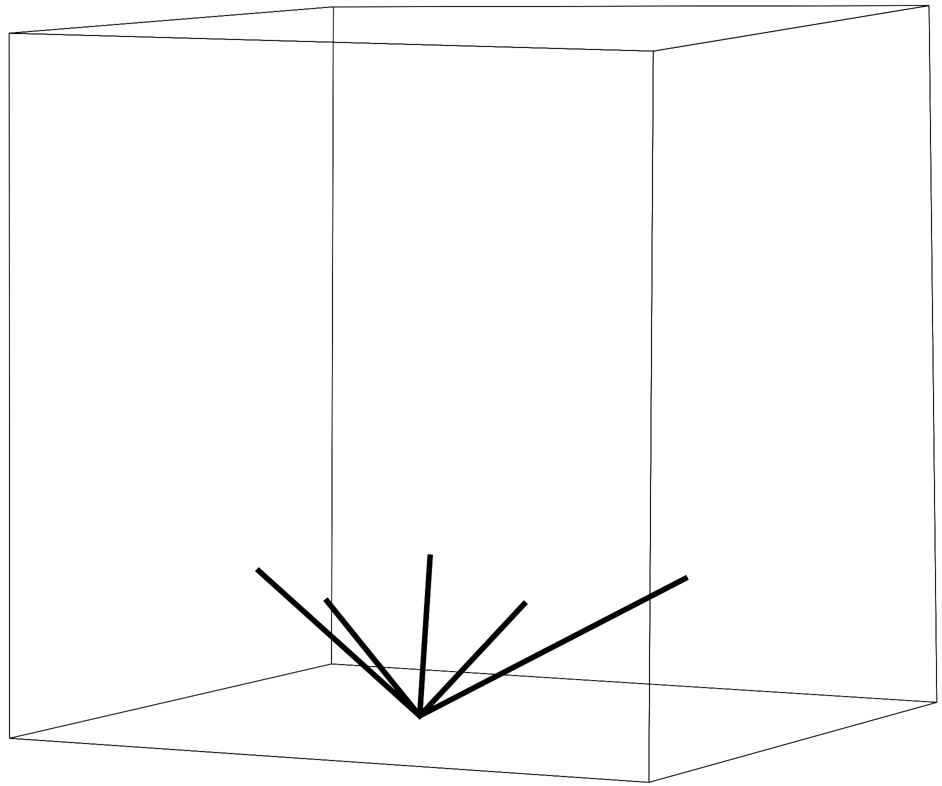
Corollary. *a-function is proportional to the volume of a zonotope:*

$$\hat{F}_P(\phi) \propto \mathrm{vol}_3(\mathcal{Z}_P(\phi)),$$
$$\mathcal{Z}_P(\phi) := \mathcal{Z}(\phi^1 \boldsymbol{v}_1, \phi^2 \boldsymbol{v}_2, \cdots, \phi^n \boldsymbol{v}_n)$$

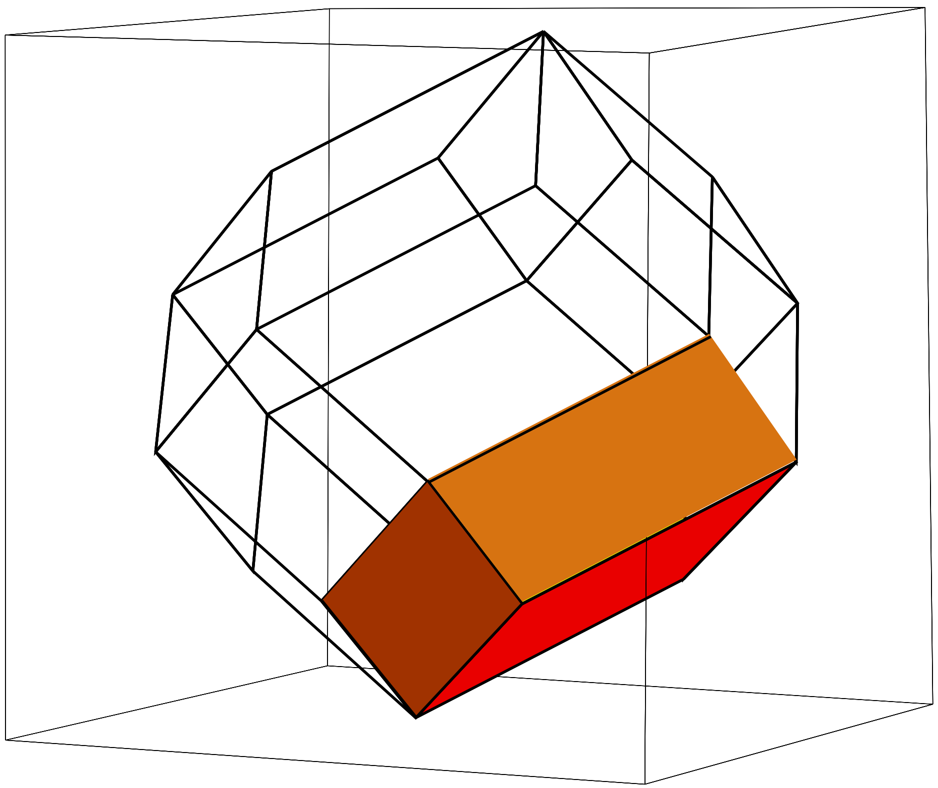
Zonotope



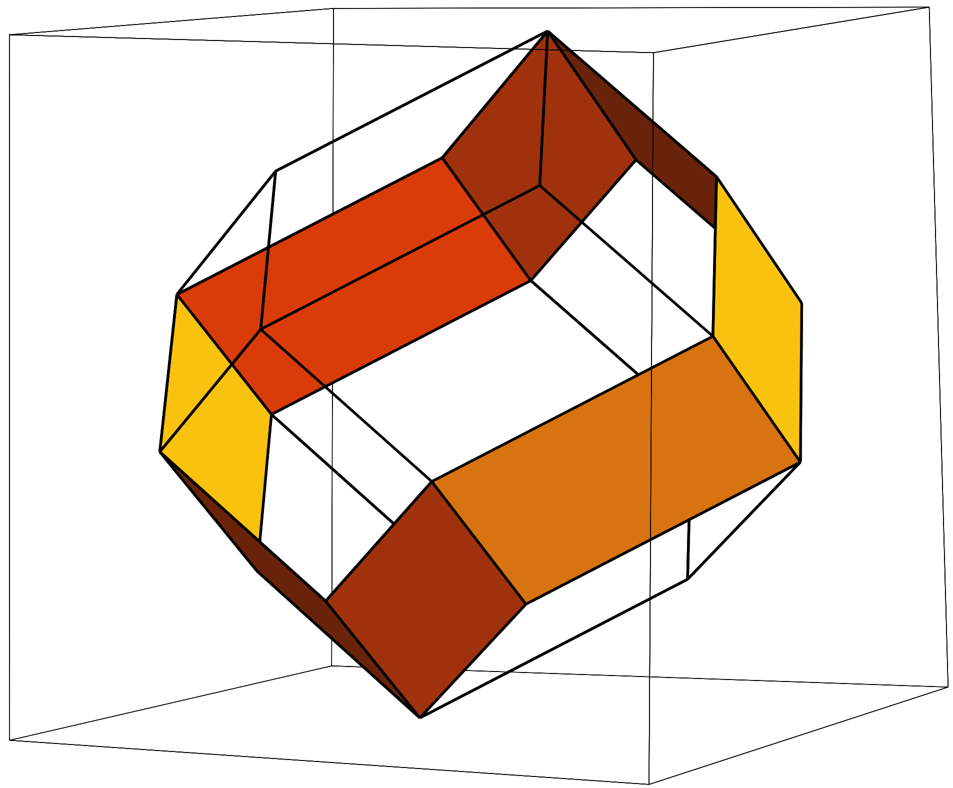
generators



cube



zone



Brunn-Minkowski inequality

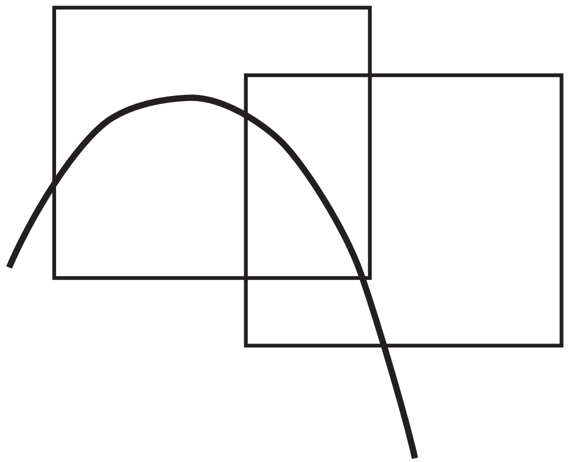
Theorem. $(\mathrm{vol}_d(-))^{1/d}$ *is concave on the set of d-dimensional bodies. Namely, if $0 \leq \lambda \leq 1$ and $A, B \subset \mathbb{R}^d$ are convex bodies, then*

$$(\mathrm{vol}_d((1-\lambda)A + \lambda B))^{1/d} \geq (1-\lambda)\,(\mathrm{vol}_d(A))^{1/d} + \lambda\,(\mathrm{vol}_d(B))^{1/d}.$$

Equality $\Longleftrightarrow$ *A and B are homothetic*

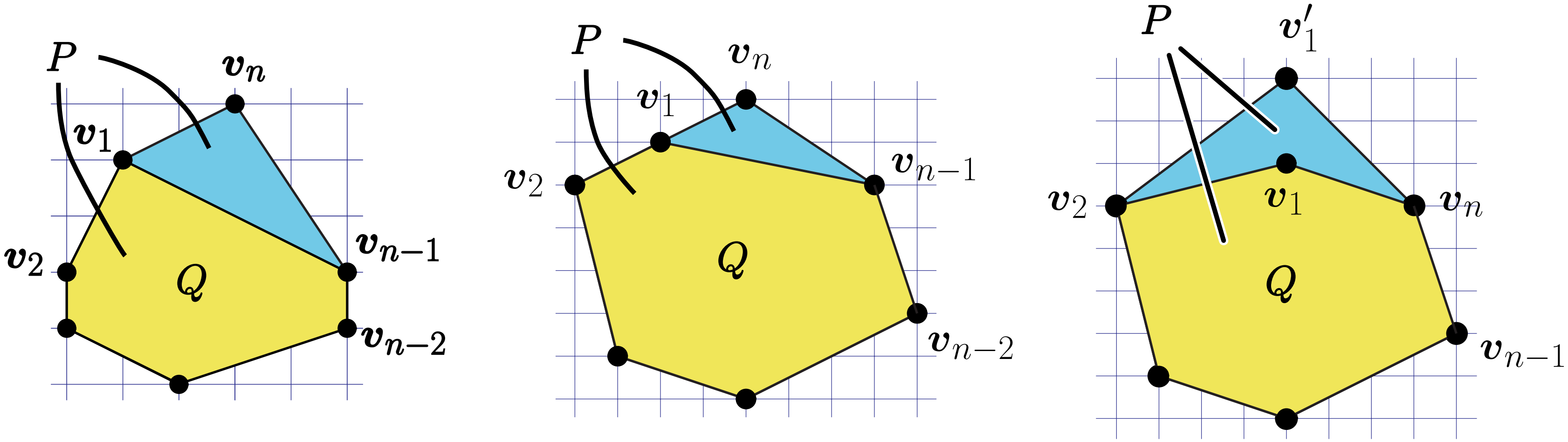
Corollary. $(F_P)^{1/3} : \Gamma_n \to \mathbb{R}$ *is strictly concave.*

Concavity + existence of critical point
$\Longrightarrow$ uniqueness & global maximum



Changing Toric Diagrams

Proposition. *Suppose toric diagrams P, Q are related as follows:*



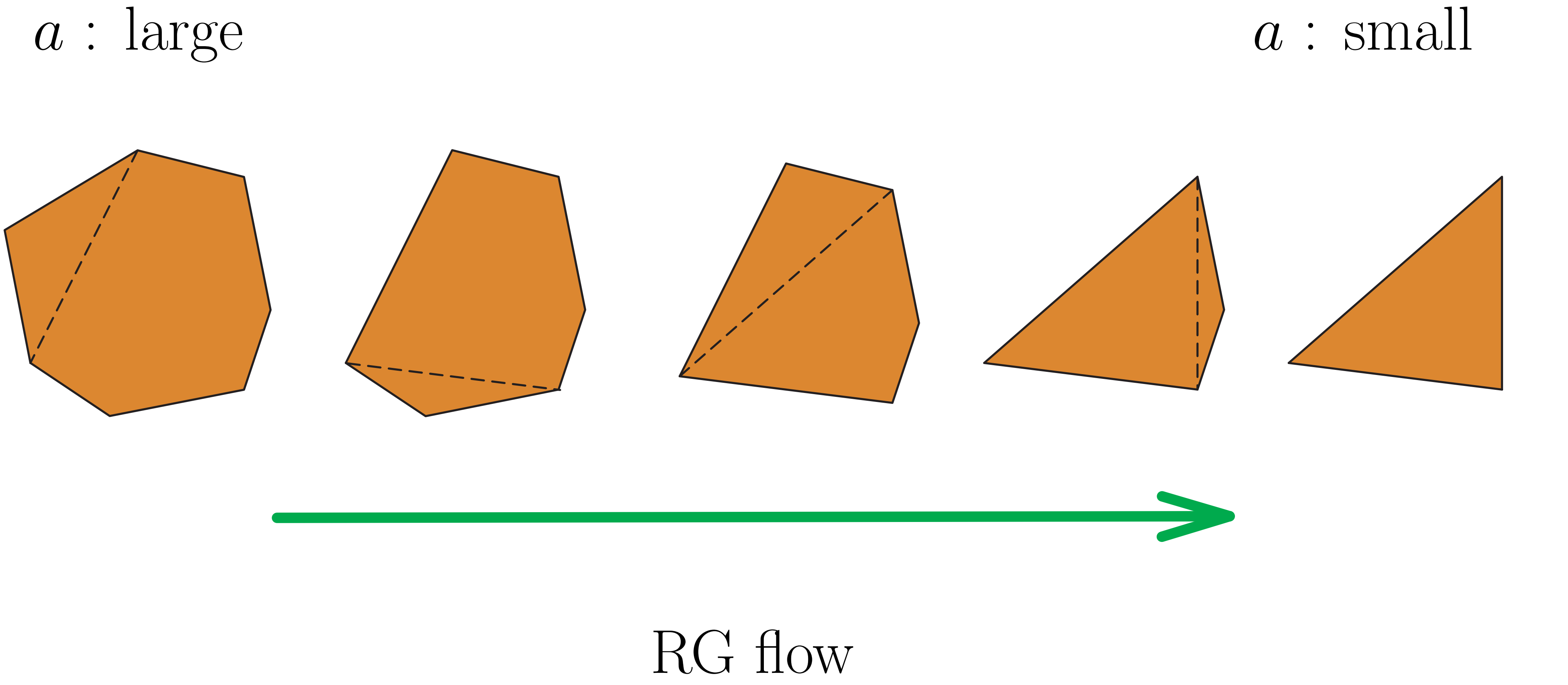
Then, $\displaystyle\max_{\phi\in\Gamma_{n-1}} F_Q(\phi) < \max_{\psi\in\Gamma_n} F_P(\psi)$*.*

$F_P : \Gamma_n \to \mathbb{R}$ cannot attain its maximum on the boundary $\partial\Gamma_n$.

$\cdots$ + continuity + strict concavity of $(F_P)^{1/3}$

$\Longrightarrow$ Existence & uniqueness of local maximum

$\Longrightarrow$ Monotinicity of $\mathfrak{M}(P)$.


a : large
a : small
RG flow

Bounds on critical points

Proposition. *If $\phi_* \in \Gamma_n$ is the critical point of F_P, then*

$$\phi_*^p = \frac{r}{3} \cdot \frac{\mathrm{vol}(\mathcal{Z}_P^{[p]}(\phi))}{\mathrm{vol}(\mathcal{Z}_P(\phi))}, \qquad (p = 1, \ldots, n).$$

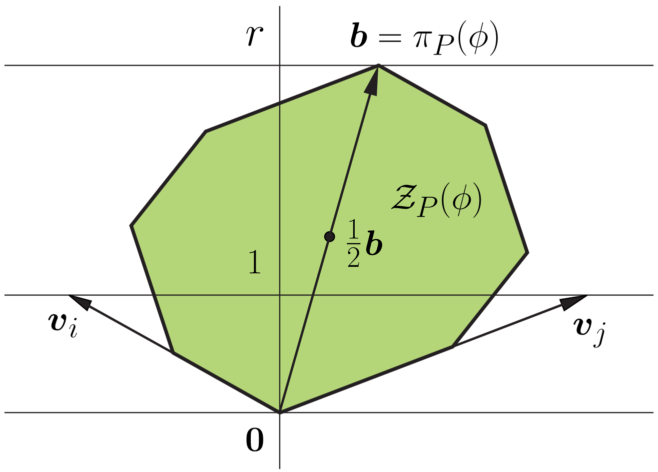
where $\mathcal{Z}_P^{[p]}(\phi)$ denotes the union of those cubes which has at least one face belonging to p-th zone.

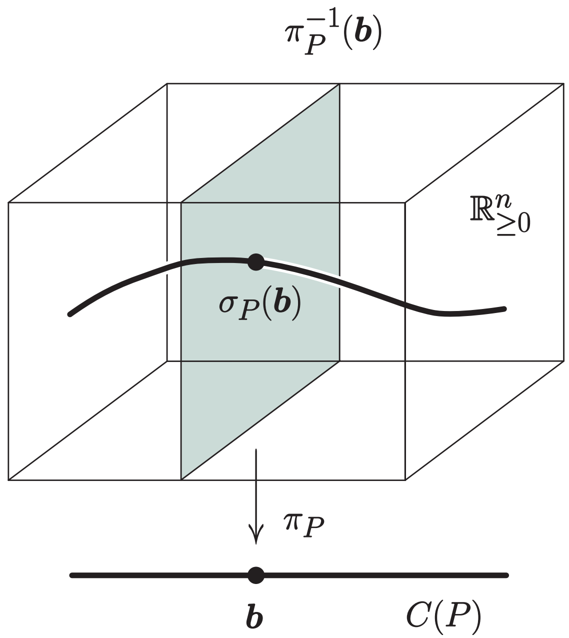
Baryonic & flavor symmetry

Maximization in two steps

$$\begin{array}{ccccc} \mathbb{R}^n_{\geq 0} & \xrightarrow{\pi_P} & C(P) & \longrightarrow & \mathbb{R} \\ \cup & & \cup & & \cup \\ \Gamma_n & \xrightarrow{\pi_P} & rP & \longrightarrow & \{r\} \end{array} \qquad \boldsymbol{b} = \pi_P(\phi) = \sum_{i=1}^{n} \phi^i \boldsymbol{v}_i$$

$$\max_{\phi \in \mathbb{R}^n_{\geq 0}} F(\phi) = \max_{\boldsymbol{b} \in C(P)} \Big(\max_{\phi \in \pi^{-1}(\boldsymbol{b})} F(\phi) \Big).$$





Relation with volume minimization

Theorem. *Suppose* $\boldsymbol{b} \in rP$ *i.e.* $\boldsymbol{b} = (*, *, r)$. *Then*

- $\hat{F}_P$ *is a quadratic polynomial along the fiber* $\pi_P^{-1}(\boldsymbol{b})$.
- *In each fiber, there is a unique critical & maximum point* $\sigma_P(\boldsymbol{b})$, *determined by* $\sigma_P^i(\boldsymbol{b}) = \dfrac{r}{V_P(\boldsymbol{b})} \ell_P^i(\boldsymbol{b})$. *Here,*

$$\ell_P^i(\boldsymbol{b}) := \frac{\langle \boldsymbol{v}_{i-1}, \boldsymbol{v}_i, \boldsymbol{v}_{i+1} \rangle}{\langle \boldsymbol{b}, \boldsymbol{v}_{i-1}, \boldsymbol{v}_i \rangle \langle \boldsymbol{b}, \boldsymbol{v}_i, \boldsymbol{v}_{i+1} \rangle} \propto \mathrm{vol}(\textit{calibrated 3-cycle})$$

$$V_P(\boldsymbol{b}) := \sum_{i=1}^{n} \ell_P^i(\boldsymbol{b}) \qquad \propto \mathrm{vol}(\textit{Sasaki-Einstein mfd.})$$

- $\displaystyle \max_{\phi \in \pi_P^{-1}(\boldsymbol{b})} \hat{F}_P(\phi) = \hat{F}_P(\sigma_P(\boldsymbol{b})) = \frac{r}{V_P(\boldsymbol{b})}.$

Martelli-Sparks-Yau, Butti-Zaffaroni

Future problems

- Global structure of 4d SCFT moduli. RG flows. Seiberg dualities.
- Inverse problem: Does the critical value of a-function characterize the toric diagram?
- Relation with dimers, algae, amoebae