Topological Aspects of Gauge Theory on Noncommutative Geometry

Hajime Aoki (Saga Univ.)

The Joint Meeting of Pacific Region Particle Physics Communities
Sheraton Waikiki, Honolulu, Hawaii
10/31/2006

collaboration with
J.Nishimura(KEK), Y. Susaki(KEK),
S. Iso(KEK), T. Maeda(Saga), K. Nagao(Ibaraki)

Main references

- H. Aoki, J. Nishimura, Y. Susaki, hep-th/0602078,
- H. Aoki, J. Nishimura, Y. Susaki, hep-th/0604093,
- H. Aoki, J. Nishimura, Y. Susaki, hep-th/06*****,

Related works

- H.Aoki, S.Iso and K.Nagao, Phys.Rev.D67(2003)065018,
- H.Aoki, S.Iso and K.Nagao, Phys.Rev.D67(2003)085005,
- H.Aoki, S.Iso and K.Nagao, Nucl.Phys.B684(2004)162-182,
- H.Aoki, S.Iso, T.Maeda and K.Nagao, Phys.Rev.D71(2005)045017,

§1. Introduction

Noncommutative Geometry

- NCG is a candidate for a new regularization of the quantum field theory

Filk '96; Grosse, Klimcik, Presnajder '95

- NCG may capture some non-local nature of quantum gravity

  NCG naturally appears from string theories and matrix models

Connes, Douglas, Schwarz '97; Aoki, Ishibash, Iso, Kawai, Kitazawa, Tada '99; Seiberg, Witten '99

New property of NCFT

- UV/IR mixing Minwara, Raamsdonk, Seiberg '00
- NC solitons (new type of topological objects)

Gopakmer, Minwalla, Strominger '00, Gross, Nekrasov, Schwarz '00

We then studied the topological aspects of gauge theory on NCG.

We have 2 points to emphasize.

1) Index and topological charge in “finite” NCG

index

$\equiv$ difference between the numbers of zero-modes for the Dirac operator with $\pm$ chirality

$=$ the topological charge of the background gauge configuration

index theorem

- In the continuum theories, these issues are well established.

  Atiyah, Singer '71, Brown, Carlitz, Lee '77

  Kim, Lee, Yang '02 (on NC $\mathbf{R}^d$)

- In the finite (discretized) setups, they become nontirivial.
  In the lattice gauge theory, they are formulated by using the Ginsparg-Wilson relation.

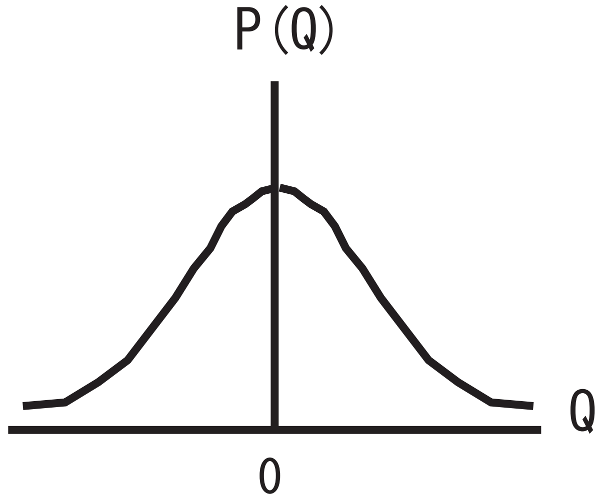
  Neuberger, Luscher, Hasenfratz, Neidermayer '98

- We define GW Dirac oprator on finite (discretized) NCG, and formulate these things.

2) We investigated “dynamics” of topological aspects

The result:

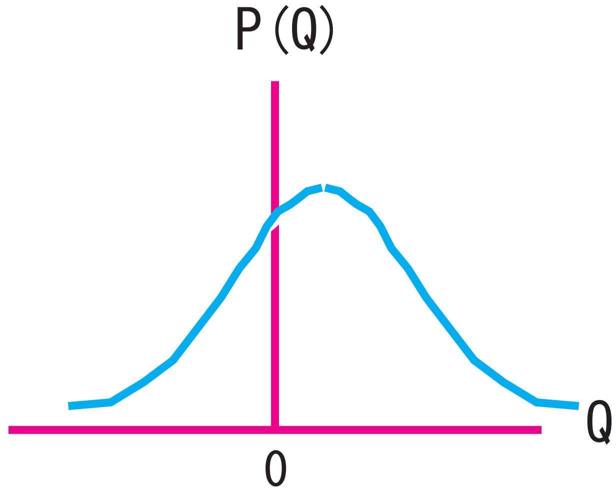
Probability distribution of the topological charge (or the index) drastically changes from the ordinary quantam field theories.



Ordinary QFT

finite width, symmetric

-->



NCFT

delta function or asymmetric under $Q \leftrightarrow -Q$

Suggestion to Phenomenology

1. If nontrivial index is generated dynamically as in the asym. distri. of the blue curve
   we may realize chiral fermion in our 4-dimensional world in string theories
   by using this phenomenon in the extra-dimensions, since

   $$\gamma^M D^M = \gamma^\mu D^\mu + \gamma^a D^a \longrightarrow \gamma^\mu D^\mu + m_4$$

   where $\gamma^a D^a \psi = m_4 \psi$.

   Dynamically from nonperturbative definition of string theory, for example,
   IIB Matrix Model (next page).

2. We may also use it for generation of baryon asymmetry of the universe,
   since non-trivial index provides $< \psi > \sim e^{-S_{\text{instanton}}} \neq 0$. 't Hooft '76

3. If only trivial sector remains as in the delta function distri. of the pink curve
   we may use it for a solution to strong CP problem.
   no instanton transition $\Rightarrow$ no need to consider θ vacuum
   no observables depend on θ of $S_\theta = i\theta Q$

IIB Matrix Model Ishibashi, Kawai, Kitazawa, Tsuchiya '96

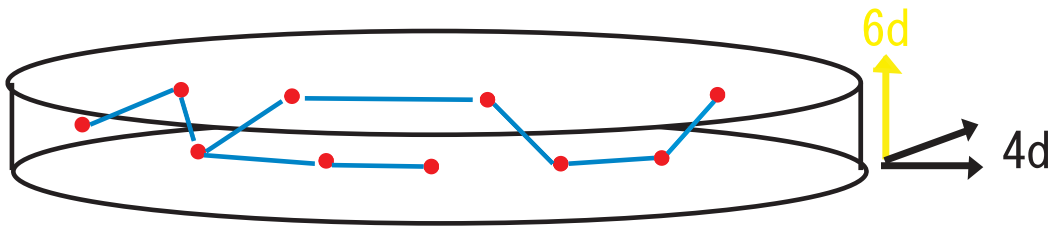
$$S_{\rm IIBMM} = -\frac{1}{g^2} Tr\Big(\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu, \psi]\Big)$$

- Proposed as a nonpertubative definition of IIB superstring theory.
- Reduced model from 10-dim. $SU(N)$ SYM to 0-dim.
- Dynamical generation of 4-dimensional Spacetime.

  Spacetime $=$ Eigenvalue distribution of A_μ.

  Aoki, Ishibashi, Kawai, Kitazawa, Tada '98

  Nishimura, Sugino '01; Kawai, Kawamoto, Kuroki, Matsuo, Shinohara, Aoyama, Shibusa '02-



plan of the talk

§2. Review on NC Torus

Coordinate and derivative on NC continuous plane

$$
\begin{aligned}
[\hat{x}_\mu \, , \, \hat{x}_\nu] &= i\,\theta_{\mu\nu} \\
\left[\hat{\partial}_\mu \, , \, \hat{x}_\nu\right] &= \delta_{\mu\nu} \\
\left[\hat{\partial}_\mu \, , \, \hat{\partial}_\nu\right] &= i\,c_{\mu\nu}
\end{aligned}
$$

On NC discretized torus (or NC periodic lattice)

$$
Z_\mu \sim e^{2\pi i \hat{x}_\mu / La} \, , \;\; \Gamma_\mu \sim e^{a\hat{\partial}_\mu}
$$

finite unitary matrices, a: lattice spacing, L: number of sites.

$$
\begin{aligned}
Z_\mu Z_\nu &= e^{2\pi i \Theta_{\mu\nu}} \, Z_\nu Z_\mu \\
\Gamma_\mu \, Z_\nu \, \Gamma_\mu^\dagger &= e^{2\pi i \delta_{\mu\nu}/L} \, Z_\nu \\
\Gamma_\mu \Gamma_\nu &= \mathcal{Z}_{\mu\nu} \Gamma_\nu \Gamma_\mu \qquad (\mathcal{Z}_{\mu\nu} = e^{2\pi i C_{\mu\nu}})
\end{aligned}
$$

Explicite construction

$$\begin{aligned}
Z_{2j-1} &= \overbrace{\mathbb{1}_L \otimes \cdots \otimes (Q_L)^{\gamma} \otimes \cdots \otimes \mathbb{1}_L}^{d/2} \\
Z_{2j} &= \mathbb{1}_L \otimes \cdots \otimes (P_L)^{-\delta} \otimes \cdots \otimes \mathbb{1}_L \\
\Gamma_{2j-1} &= \mathbb{1}_L \otimes \cdots \otimes (P_L)^{\alpha} \otimes \cdots \otimes \mathbb{1}_L \\
\Gamma_{2j} &= \mathbb{1}_L \otimes \cdots \otimes (Q_L)^{\beta} \otimes \cdots \otimes \mathbb{1}_L
\end{aligned}$$

where the dimension d is assumed to be even,

$$P_L = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix} \quad , \quad Q_L = \begin{pmatrix} 1 & & & & \\ & e^{2\pi i/L} & & & \\ & & e^{4\pi i/L} & & \\ & & & \ddots & \\ & & & & e^{2\pi i(L-1)/L} \end{pmatrix}$$

are the 't Hooft matrices, and satisfy $P_L Q_L = e^{2\pi i/L} Q_L P_L$.
$\alpha, \beta, \gamma, \delta$ are integeres which determine $\Theta_{\mu\nu}$ and $C_{\mu\nu}$.

Functions ↔ Matrices

$$F(x) = \sum_{\vec{m} \in (\mathbf{Z}^d)_L} f_{\vec{m}} e^{2\pi i \vec{m} \cdot \vec{x}/La} \longleftrightarrow \hat{F} = \sum_{\vec{m} \in (\mathbf{Z}^d)_L} f_{\vec{m}} \hat{\phi}_{\vec{m}}$$

where

$$\begin{aligned} \hat{\phi}_{\vec{m}} &= (Z_1)^{m_1} \cdots (Z_d)^{m_d} e^{\pi i \sum_{\mu<\nu} \Theta_{\mu\nu} m_\mu m_\nu} \\ &\sim e^{2\pi i \vec{m} \cdot \vec{\hat{x}}/La} \end{aligned}$$

Gauge theory on discretized NC torus

Gonzalez-Arroyo, Korthals Altas '83, Ambjorn,Makeenko,Nishimura,Szabo '99; Griguolo,Seminara '03

• Gauge action in terms of fields on the NC lattice

$$S_{\text{lattice}} = -\beta \sum_x \sum_{\mu\neq\nu} U_\mu(x) \star U_\nu(x+a\hat{\mu}) \star U_\mu(x+a\hat{\nu})^* \star U_\nu(x)^*$$

$$U_\mu(x) \star U_\mu(x)^* = U_\mu(x)^* \star U_\mu(x) = 1$$

$U_\mu(x)$ are not unitary, but star unitary.

• $\Leftrightarrow$ Gauge action in terms of matrices

$$\begin{aligned} S_{\text{matrix}} &= -N\beta \sum_{\mu\neq\nu} \operatorname{tr}\left\{U_\mu\left(\Gamma_\mu U_\nu \Gamma_\mu^\dagger\right)\left(\Gamma_\nu U_\mu^\dagger \Gamma_\nu^\dagger\right) U_\nu^\dagger\right\} + 2\beta N^2 \\ &= -N\beta \sum_{\mu\neq\nu} \mathcal{Z}_{\nu\mu} \operatorname{tr}\left(V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger\right) + 2\beta N^2 \end{aligned}$$

where $V_\mu \equiv U_\mu \Gamma_\mu$ are unitary matrices.
Nothing but Twisted Eguchi-Kawai model Gonzalez-Arroyo, Okawa '83; Eguchi, Nakayama'83.

Ginsparg-Wilson Dirac operator Nishimura,Vazquez-Mozo '01; Aoki,Iso,Nagao '02

$$D_{\rm GW} = \frac{1}{a}(1 - \Gamma\hat{\Gamma})$$

where Neuberger '98

$$\begin{aligned} \Gamma &= \gamma_5 \\ \hat{\Gamma} &= \frac{H}{\sqrt{H^2}} \\ H &= \gamma_5 \left(1 - a D_{\rm W}\right) \end{aligned}$$

where $D_{\rm W}$ is the Wilson Dirac operator

$$\begin{aligned} D_{\rm W} &= \frac{1}{2}\sum_{\mu=1}^{d}\left\{\gamma_\mu\left(\nabla_\mu^* + \nabla_\mu\right) - a\nabla_\mu^*\nabla_\mu\right\} \\ \nabla_\mu\Psi &= \frac{1}{a}\left[U_\mu(\Gamma_\mu\Psi\Gamma_\mu^\dagger) - \Psi\right] = \frac{1}{a}\left[V_\mu\Psi\Gamma_\mu^\dagger - \Psi\right], \\ \nabla_\mu^*\Psi &= \frac{1}{a}\left[\Psi - (\Gamma_\mu^\dagger U_\mu^\dagger\Gamma_\mu)(\Gamma_\mu^\dagger\Psi\Gamma_\mu)\right] = \frac{1}{a}\left[\Psi - V_\mu^\dagger\Psi\Gamma_\mu\right] \end{aligned}$$

GW relation is satisfied by the definition.

$$
\begin{aligned}
\Gamma D_{\mathrm{GW}} + D_{\mathrm{GW}} \hat{\Gamma} &= 0 \\
(\Leftrightarrow \Gamma D_{\mathrm{GW}} + D_{\mathrm{GW}} \Gamma &= a D_{\mathrm{GW}} \Gamma D_{\mathrm{GW}})
\end{aligned}
$$

Index of the GW Dirac operator

$$
\mathrm{Index}(D_{\mathrm{GW}}) \equiv n_+ - n_- = \frac{1}{2} \mathcal{T}r(\Gamma + \hat{\Gamma})
$$

where $n_\pm$ is the number of zero-modes with $\pm$ chirality.
$\mathcal{T}r$ represents a trace over the space of matrix and the spinor index.

1. $\frac{1}{2}\mathcal{T}r(\Gamma + \hat{\Gamma})$ takes only integer values, since Γ and $\hat{\Gamma}$ have the form of sign operator.

2. In the continuum limit, it takes the form of the topological charge. Iso, Nagao '02

3. For topologically nontrivial gauge backgrounds, it takes the corresponding nonzero values.

Topological charge on finite NCG

$$\begin{aligned} Q &= \frac{1}{4\pi i} \sum_x \sum_{\mu\nu} \epsilon_{\mu\nu} U_\mu(x) \star U_\nu(x + a\hat{\mu}) \star U_\mu(x + a\hat{\nu})^* \star U_\nu(x)^* \ , \\ &= \frac{1}{4\pi i} N \sum_{\mu\nu} \epsilon_{\mu\nu} \mathcal{Z}_{\nu\mu} \mathrm{tr} \left(V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger \right) \end{aligned}$$

A naive discretization of the one in the continuum theory.
Q takes non-integer values in general.

cf) A more sophisticated definition are given in lattice theory. Hasenfratz '98

§3. Analysis with classical solutions

Aoki,Nishimura,Susaki, hep-th/0602078

General classical solutions van Baal '83, Griguolo,Seminara '03

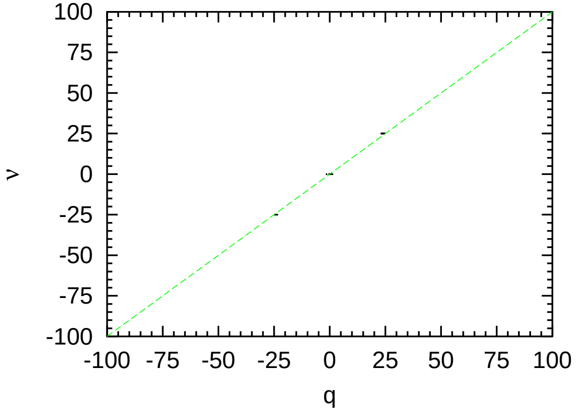
$$V_\mu = \begin{pmatrix} \Gamma_\mu^{(1)} & & & \\ & \Gamma_\mu^{(2)} & & \\ & & \ddots & \\ & & & \Gamma_\mu^{(k)} \end{pmatrix}$$

where

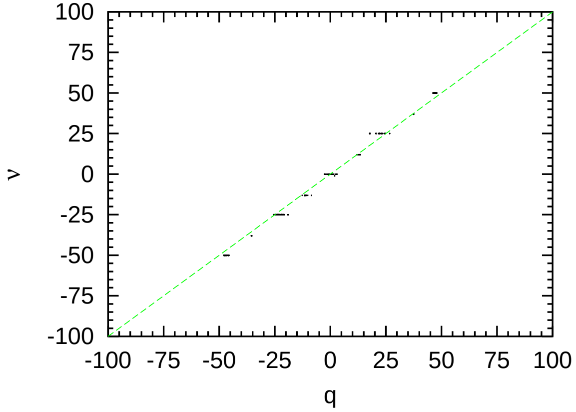
$$\begin{aligned} \Gamma_\mu^{(j)} \Gamma_\nu^{(j)} &= Z_{\mu\nu}^{(j)} \Gamma_\nu^{(j)} \Gamma_\mu^{(j)} \\ Z_{12}^{(j)} &= Z_{21}^{(j)*} = \exp\left(2\pi i \frac{m_j}{n_j}\right) \end{aligned}$$

Index vs Topological charge (Scatter plot for classical solutions with small action)

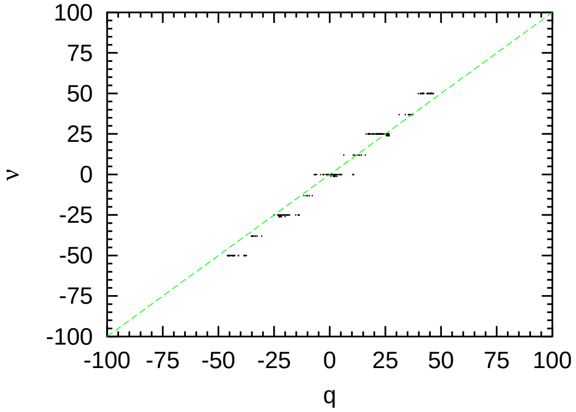
$\nu \simeq Q \Rightarrow$ index theorem $\qquad N = 25$



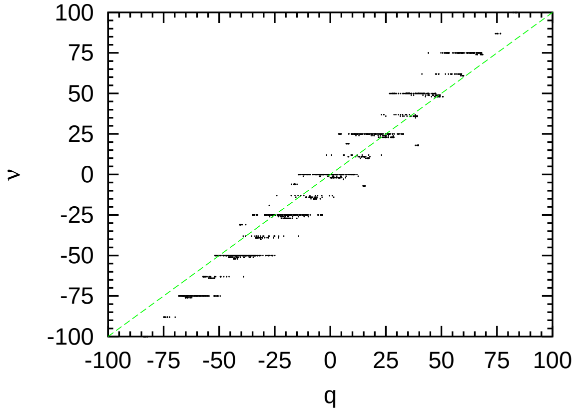
$S/\beta < 100$



$S/\beta < 200$



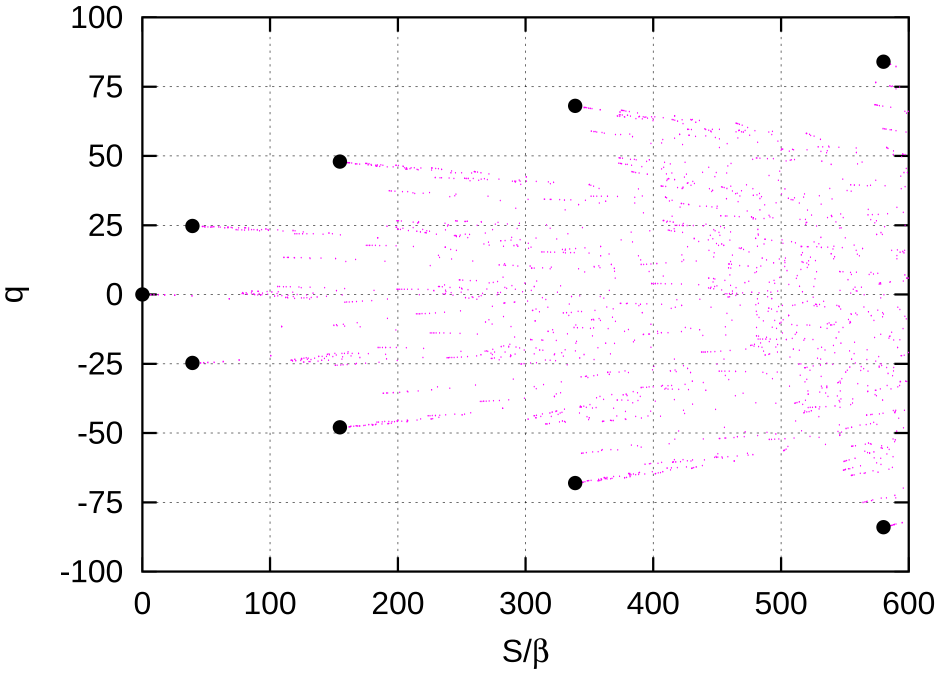
$S/\beta < 300$



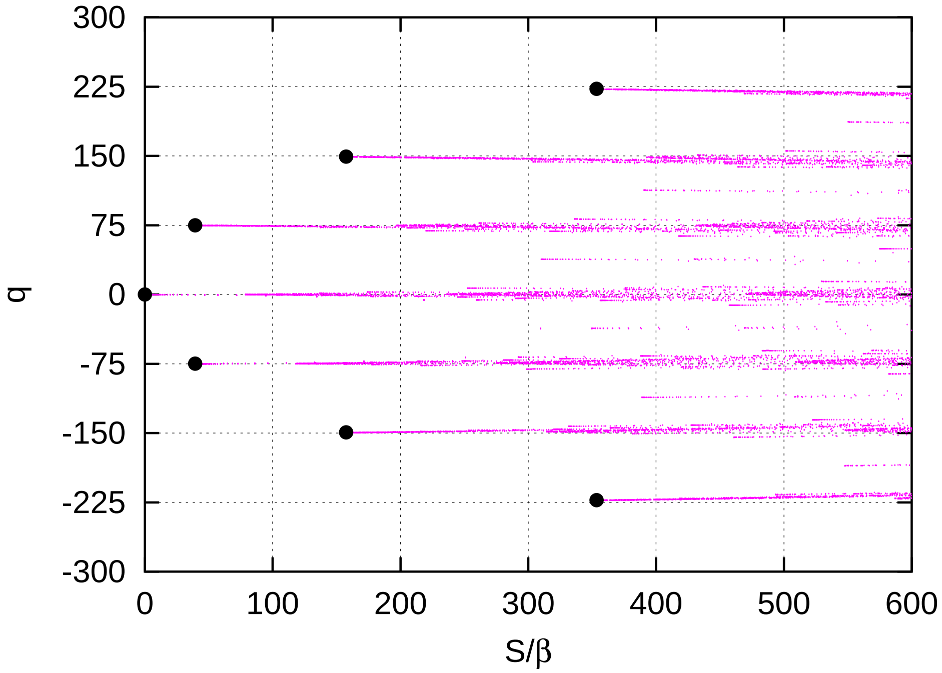
$S/\beta < 600$

Topological charge vs action

Scatter plot for the classical solutions (black blobs are of one-block-solutions)



$N = 25$



$N = 75$

• For the classical solutions

$$
\begin{aligned}
S &= 4N\beta \sum_{1\le j\le k} n_j \sin^2\left\{\pi\left(\frac{m_j}{n_j}-\frac{M}{N}\right)\right\} \\
Q &= \frac{N}{2\pi} \sum_{1\le j\le k} n_j \sin\left\{2\pi\left(\frac{m_j}{n_j}-\frac{M}{N}\right)\right\}
\end{aligned}
$$

• In $N \to \infty$, for finite action configurations,

$$
Q \simeq N\left(\sum_j m_j - M\right)
$$

Q takes only multiple integers of N.

Minimum action is given by one-block-solution

$$
S \simeq 4\pi^2\beta\left(\frac{Q}{N}\right)^2
$$

• In $\beta \to \infty$, only $Q = 0$ sector servives.

Comparison to the Commutative case

- General classical solutions (up to moduli)

$$
\begin{aligned}
U_1(x) &= \begin{cases} 1 & \text{if } x_1 \neq a(N-1) \\ \exp(-2\pi i x_2 Q/aN) & \text{if } x_1 = a(N-1) \end{cases} \\
U_2(x) &= \exp(2\pi i x_1 Q/aN^2)
\end{aligned}
$$

with topological charge Q (arbitrary integer values)

$$S_{\text{classical}}(Q) \simeq \frac{4\pi^2\beta}{N^2} Q^2$$

- When Q is a multiple integer of N
  $\Rightarrow$ Translatoinally invariant in 2nd direction $\Rightarrow$ Star product becomes ordinary product
  $\Rightarrow$ The above configurations are the classical solutions in NC theory as well.

Giusti, Gonzalez-Arroyo, Hoelbling, Neuberger, Rebbi '02, Kiskis, Narayanan, Neuberger '02

- For the other cases, situations change since singularity is not allowed in the NCG.

§4. Analysis by Monte Carlo simulation

Aoki,Nishimura,Susaki, hep-th/0604093

Model and Observable (again)

- Model

$$S_{\mathrm{matrix}} = -N\beta \sum_{\mu\neq\nu} \mathcal{Z}_{\nu\mu} \mathrm{tr}\left(V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger\right) + 2\beta N^2$$

- Observe

$$\nu \equiv \mathrm{Index}(D_{\mathrm{GW}}) = \frac{1}{2}\mathcal{T}r(\Gamma + \hat{\Gamma}) = \frac{1}{2}\mathcal{T}r\frac{H}{\sqrt{H^2}}$$

  where

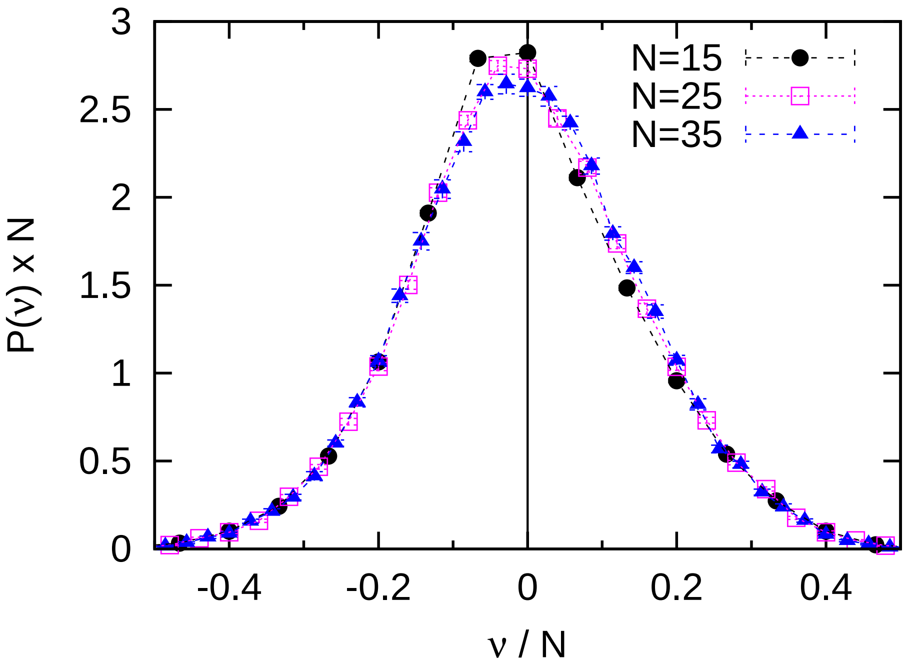
$$H = \gamma_5\left(1 - aD_{\mathrm{W}}\right)$$

- Obtain the probability distribution (or histgram) of the index ν.

Probability distribution of index $P(\nu)$

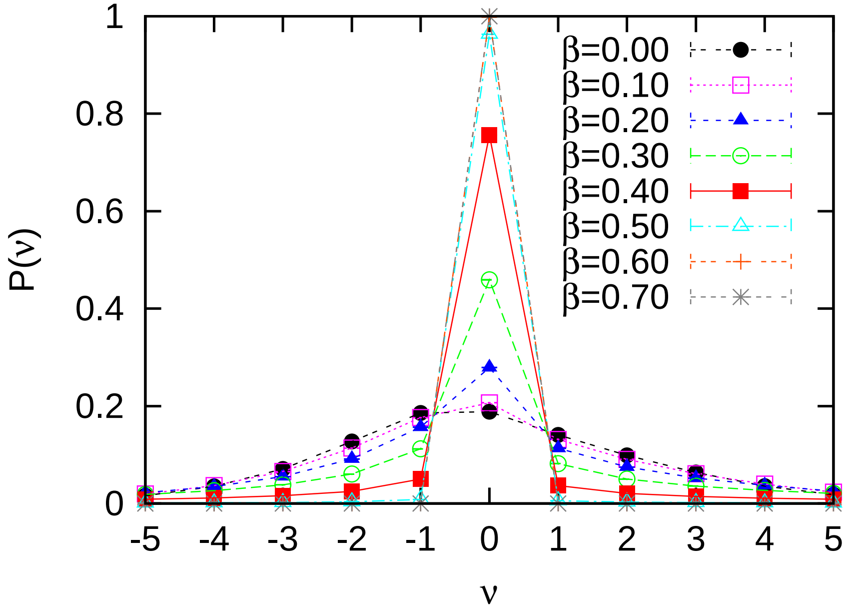
• $\beta = 0$

ie, without taking acount of the action

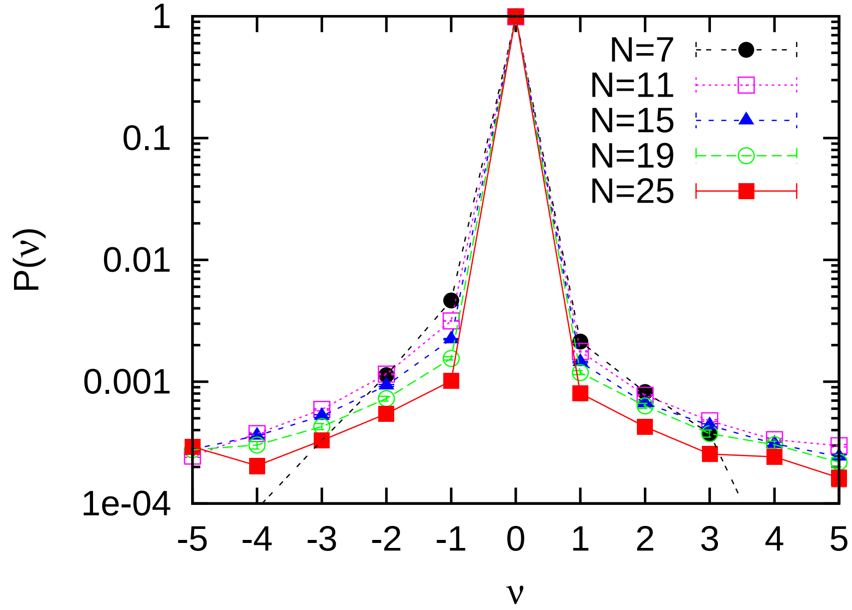


- Topologically nontrivial configurations with finite probability.
- Same distribution of ν/N, for various N.
- $P(\nu)$ is asymmetric under $\nu \leftrightarrow -\nu$. $\Rightarrow$ Generation of index.
  We can expect it quite generically from the parity violation due to the NCG.

• $\beta \neq 0$



various β at $N = 15$

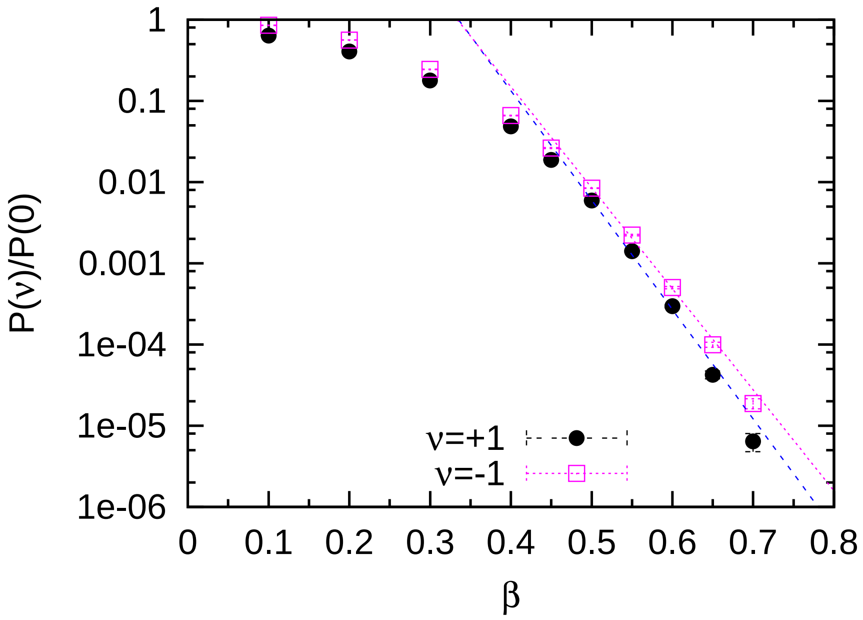


various N at $\beta = 0.55$

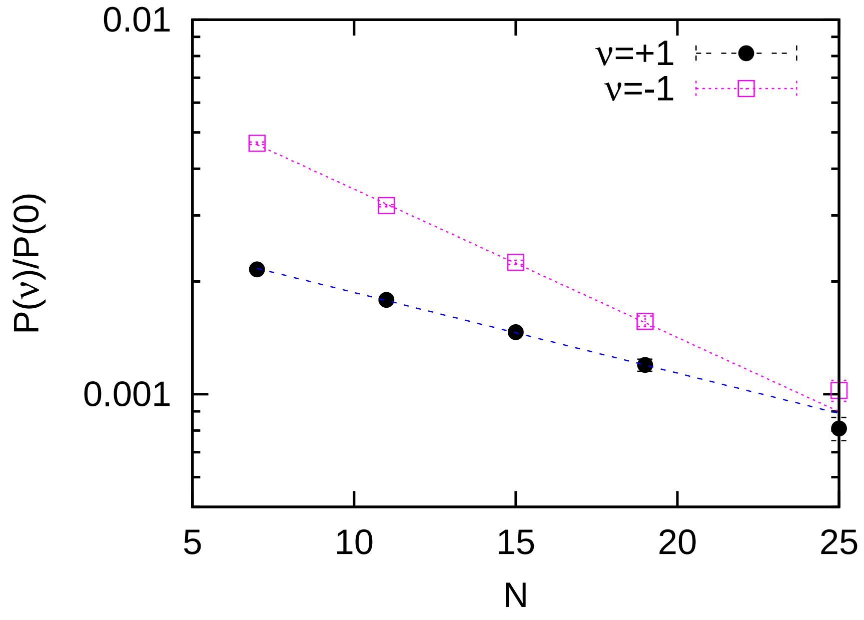
- $P(\nu) \to \delta_{\nu 0}$ for increasing β and for increasing N.
- Continuum limit: $N, \beta \to \infty$ with $\theta = N/\beta$ fixed. Bietenholz, Hofheinz,Nishimura '02
- The above behavior is consistent with the instanton calculus in the continuum theory.

Paniak, Szabo '02

$P(\nu)/P(0)$ for $\nu = \pm 1$ *versus* β and N



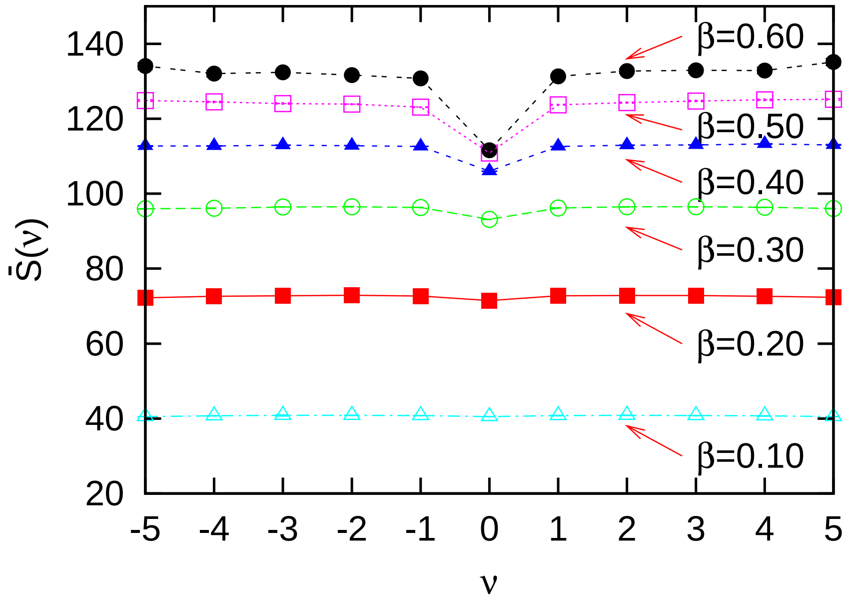
various β at $N = 15$



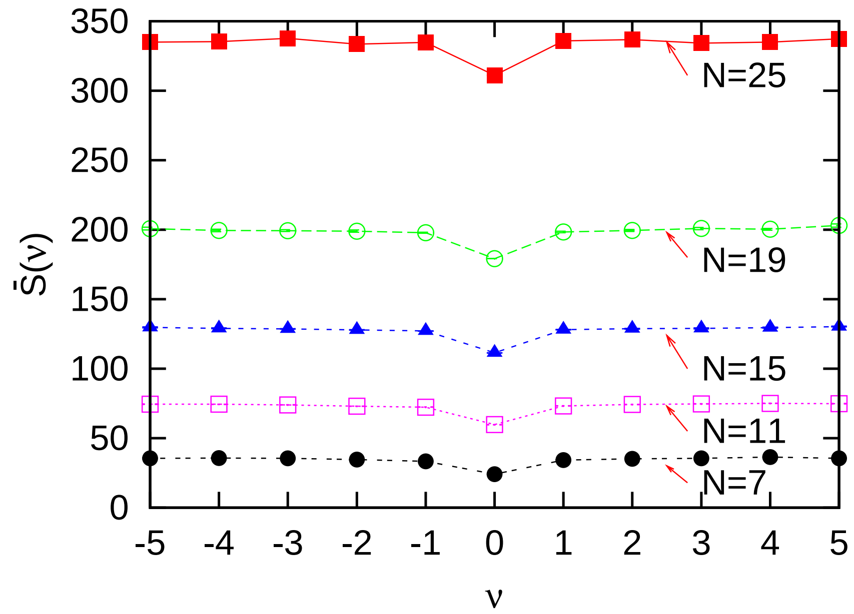
various N at $\beta = 0.55$

Exponentially decreasing behavior of $P(\nu)/P(0)$ as a function of β and of N.

Average action in each topological sector



various β at $N = 15$

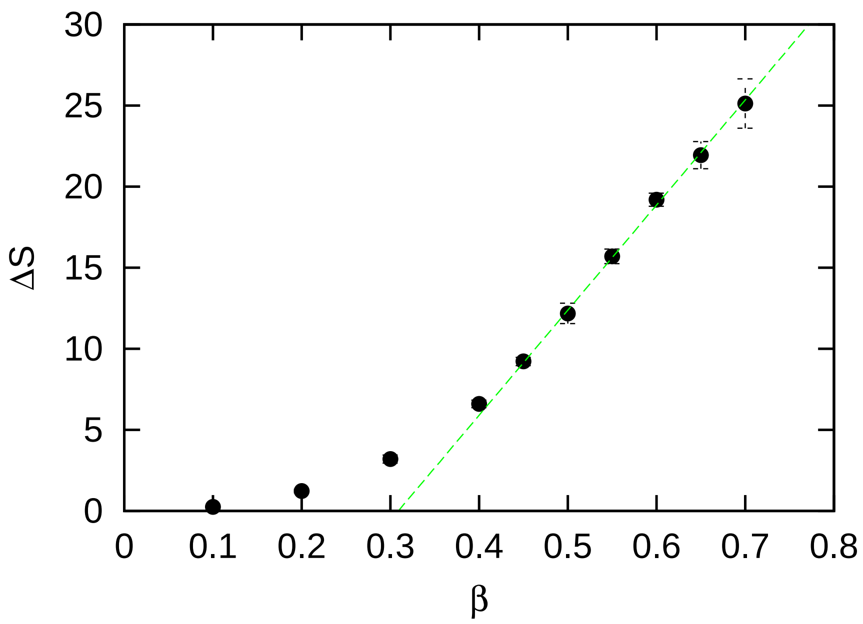


various N at $\beta = 0.55$

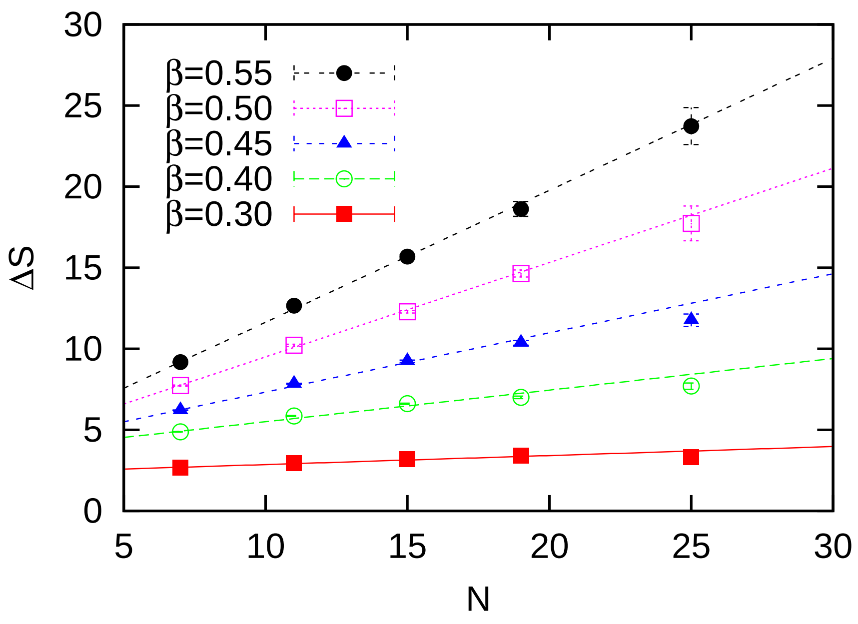
A dip at $\nu = 0$ and platteau for $\nu \neq 0$

Depth of the dip

$$\Delta S \equiv \bar{S}(\nu = -1) - \bar{S}(\nu = 0)$$



ΔS vs β for $N = 15$



ΔS vs N for various β

Linear dependence $\Rightarrow$

Consistent with the exponentially decreasing behavior of $P(\nu)/P(0)$.

Consistent with the analysis about the classical solutions.

Nontrivial projective modules

Aoki,Nishimura,Susaki, hep-th/06*****

- So far, we have considered in the trivial module.

- Topologically nontrivial sectors in the gauge theory on the discretized NC torus can be constructed by using the projective module in the twisted reduced model, usually labeled by two integeres (p, q). Griguolo,Seminara '03

$$S = -N\beta \sum_{\mu \neq \nu} \mathcal{Z}_{\nu\mu} \mathrm{tr} \left(V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger \right) + 2\beta N^2$$

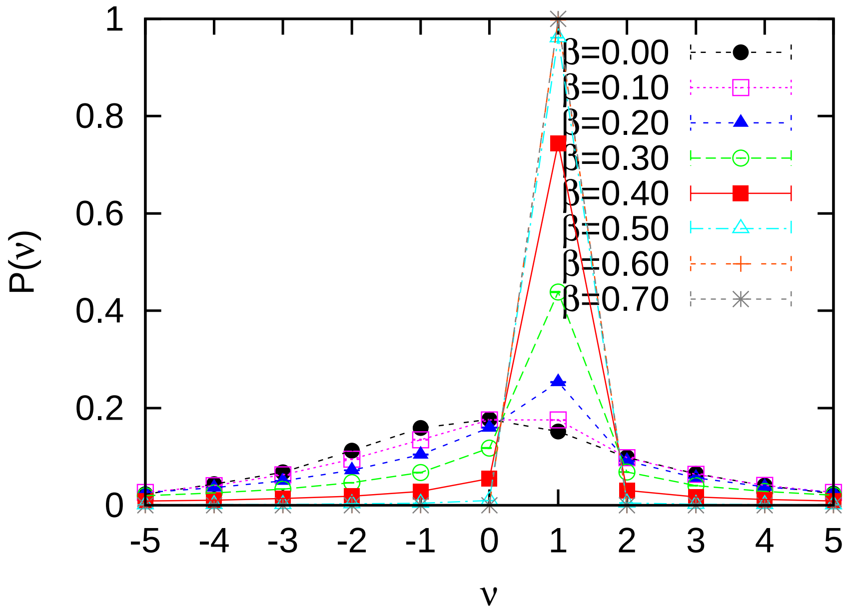
Matrix size N and the twist $\mathcal{Z}_{\mu\nu}$ are tuned by the integers p, q.
For the small action configurations, the topological charge Q takes the value

$$Q = q$$

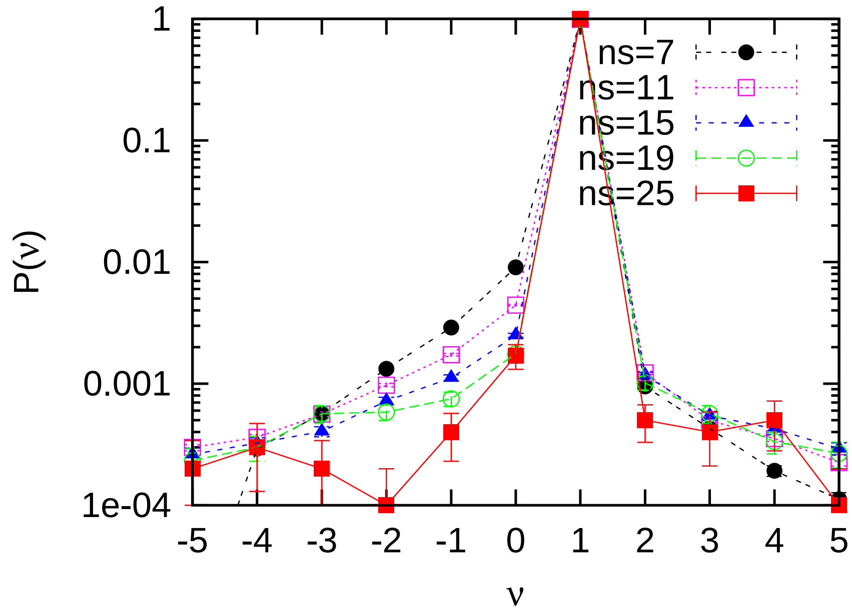
Preliminary results of Monte Carlo simulations

Probability distribution of the index ν

$q = 1$



various β at $N = 15$



various N at $\beta = 0.55$

- $P(\nu) \to \delta_{\nu 1}$ for increasing β and for increasing N.
- We also observed a dip at $\nu = 1$ in the average action.

§5. Conclusion and discussion

- GW relation is useful for defining the index in NCG or MM.
- We found new dynamical aspects of topology perculiar to NCG.
  - Generation of nontrivial index
  - A single sector survives the continuum limit

  $\Rightarrow$ chiral fermion in our 4-dimensional world in string theories
  baryogenesis, solution to strong CP problem
- Extension to 4 and higher dimensions such as $\mathbf{T}^4$, $\mathbf{S}^2 \times \mathbf{S}^2$, $\mathbf{CP}^2$.
- Extension to the study of the topology of spacetime.
- Study about how to embed (curved) space and matter in MM.