Topological Aspects of Gauge Theory on Noncommutative Geometry

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collaboration with
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Main references
• H. Aoki, J. Nishimura, Y. Susaki, hep-th/0602078,
• H. Aoki, J. Nishimura, Y. Susaki, hep-th/0604093,
• H. Aoki, J. Nishimura, Y. Susaki, hep-th/06*****,

Related works
§1. Introduction

Noncommutative Geometry

- NCG is a candidate for a new regularization of the quantum field theory
  Filk '96; Grosse, Klimcik, Presnajder '95
- NCG may capture some non-local nature of quantum gravity
  NCG naturally appears from string theories and matrix models
  Connes, Douglas, Schwarz '97; Aoki, Ishibash, Iso, Kawai, Kitazawa, Tada '99; Seiberg, Witten '99

New property of NCFT

- UV/IR mixing Minwara, Raamsdonk, Seiberg '00
- NC solitons (new type of topological objects)
  Gopakmer, Minwalla, Strominger '00, Gross, Nekrasov, Schwarz '00

We then studied the topological aspects of gauge theory on NCG.
We have 2 points to emphasize.
1) Index and topological charge in “finite” NCG

Index
≡ difference between the numbers of zero-modes for the Dirac operator with ± chirality
= the topological charge of the background gauge configuration
index theorem

• In the continuum theories, these issues are well established.
  Atiyah, Singer ’71, Brown, Carlitz, Lee ’77
  Kim, Lee, Yang ’02 (on NC $\mathbb{R}^d$)

• In the finite (discretized) setups, they become nontrivial.
  In the lattice gauge theory, they are formulated by using the Ginsparg-Wilson relation.
  Neuberger, Luscher, Hasenfratz, Neidermayer ’98

• We define GW Dirac operator on finite (discretized) NCG, and formulate these things.
2) We investigated “dynamics” of topological aspects

The result:
Probability distribution of the topological charge (or the index) drastically changes from the ordinary quantum field theories.

Ordinary QFT
finite width, symmetric

NCFT
delta function or asymmetric under $Q \leftrightarrow -Q$
Suggestion to Phenomenology

1. If nontrivial index is generated dynamically as in the asym. distri. of the blue curve we may realize chiral fermion in our 4-dimensional world in string theories by using this phenomenon in the extra-dimensions, since

\[ \gamma^M D^M = \gamma^\mu D^\mu + \gamma^a D^a \rightarrow \gamma^\mu D^\mu + m_4 \]

where \( \gamma^a D^a \psi = m_4 \psi \).

Dynamically from nonperturbative definition of string theory, for example, IIB Matrix Model (next page).

2. We may also use it for generation of baryon asymmetry of the universe, since non-trivial index provides \( < \psi > \sim e^{-S_{\text{inst}} \neq 0} \). \( \text{'}t \text{ Hooft '76} \)

3. If only trivial sector remains as in the delta function distri. of the pink curve we may use it for a solution to strong CP problem.

no instanton transition \( \Rightarrow \) no need to consider \( \theta \) vacuum

no observables depend on \( \theta \) of \( S_\theta = i\theta Q \)
IIB Matrix Model Ishibashi, Kawai, Kitazawa, Tsuchiya ’96

\[ S_{IIBMM} = -\frac{1}{g^2} Tr \left( \frac{1}{4} [A_\mu, A_\nu] [A^\mu, A^\nu] + \frac{1}{2} \bar{\psi} \Gamma^\mu [A_\mu, \psi] \right) \]

- Proposed as a nonpertubative definition of IIB superstring theory.
- Reduced model from 10-dim. \( SU(N) \) SYM to 0-dim.
- Dynamical generation of 4-dimensional Spacetime.
  Spacetime = Eigenvalue distribution of \( A_\mu \).

Aoki, Ishibashi, Kawai, Kitazawa, Tada ’98

Nishimura, Sugino ’01; Kawai, Kawamoto, Kuroki, Matsuo, Shinohara, Aoyama, Shibusa ’02-
plan of the talk

§1 Introduction
§2 Review on NC torus
§3 Analysis with classical solutions
§4 Analysis by Monte Carlo simulation
§5 Conclusion and discussion
2. Review on NC Torus

Coordinate and derivative on NC continuous plane

\[
\begin{align*}
[\hat{x}_\mu, \hat{x}_\nu] &= i \theta_{\mu\nu} \\
[\hat{\partial}_\mu, \hat{x}_\nu] &= \delta_{\mu\nu} \\
[\hat{\partial}_\mu, \hat{\partial}_\nu] &= i c_{\mu\nu}
\end{align*}
\]

On NC discretized torus (or NC periodic lattice)

\[
Z_\mu \sim e^{2\pi i \hat{x}_\mu / La}, \quad \Gamma_\mu \sim e^{a \hat{\partial}_\mu}
\]

finite unitary matrices, \(a\): lattice spacing, \(L\): number of sites.

\[
\begin{align*}
Z_\mu Z_\nu &= e^{2\pi i \Theta_{\mu\nu}} Z_\nu Z_\mu \\
\Gamma_\mu Z_\nu \Gamma_\mu^\dagger &= e^{2\pi i \delta_{\mu\nu} / L} Z_\nu \\
\Gamma_\mu \Gamma_\nu &= Z_{\mu\nu} \Gamma_\nu \Gamma_\mu \quad (Z_{\mu\nu} = e^{2\pi i C_{\mu\nu}})
\end{align*}
\]
Explicit construction

\[
Z_{2j-1} = \underbrace{1_L \otimes \cdots \otimes (Q_L)^\gamma}_{d/2} \otimes \cdots \otimes 1_L
\]

\[
Z_{2j} = 1_L \otimes \cdots \otimes (P_L)^-\delta \otimes \cdots \otimes 1_L
\]

\[
\Gamma_{2j-1} = 1_L \otimes \cdots \otimes (P_L)^\alpha \otimes \cdots \otimes 1_L
\]

\[
\Gamma_{2j} = 1_L \otimes \cdots \otimes (Q_L)^\beta \otimes \cdots \otimes 1_L
\]

where the dimension \(d\) is assumed to be even,

\[
P_L = \begin{pmatrix}
0 & 1 & & & \\
0 & 1 & & & \\
& \ddots & \ddots & \ddots & \\
1 & \cdots & & & 1
\end{pmatrix}
\]

\[
Q_L = \begin{pmatrix}
1 & e^{2\pi i/L} & & \\
& e^{4\pi i/L} & & \\
& & \ddots & \\
& & & e^{2\pi i(L-1)/L}
\end{pmatrix}
\]

are the 't Hooft matrices, and satisfy \(P_L Q_L = e^{2\pi i/L} Q_L P_L\).

\(\alpha, \beta, \gamma, \delta\) are integers which determine \(\Theta_{\mu\nu}\) and \(C_{\mu\nu}\).
Functions ↔ Matrices

\[ F(x) = \sum_{\vec{m} \in (Z^d)_L} f_{\vec{m}} e^{2\pi i \vec{m} \cdot \vec{x}/La} \quad \longleftrightarrow \quad \hat{F} = \sum_{\vec{m} \in (Z^d)_L} f_{\vec{m}} \hat{\phi}_{\vec{m}} \]

where

\[ \hat{\phi}_{\vec{m}} = (Z_1)^{m_1} \cdots (Z_d)^{m_d} e^{\pi i \sum_{\mu<\nu} \Theta_{\mu\nu} m_\mu m_\nu} \]

\[ \sim e^{2\pi i \vec{m} \cdot \vec{x}/La} \]
Gauge theory on discretized NC torus

Gonzalez-Arroyo, Korthals Altas '83, Ambjorn, Makeenko, Nishimura, Szabo '99; Griguolo, Seminara '03

• Gauge action in terms of fields on the NC lattice

\[ S_{\text{lattice}} = -\beta \sum_x \sum_{\mu \neq \nu} U_\mu(x) \star U_\nu(x + a\hat{\mu}) \star U_\mu(x + a\hat{\nu})^* \star U_\nu(x)^* \]

\[ U_\mu(x) \star U_\mu(x)^* = U_\mu(x)^* \star U_\mu(x) = 1 \]

\( U_\mu(x) \) are not unitary, but star unitary.

• ⇔ Gauge action in terms of matrices

\[ S_{\text{matrix}} = -N\beta \sum_{\mu \neq \nu} \text{tr} \left\{ U_\mu (\Gamma_\mu U_\nu \Gamma_\mu^\dagger) (\Gamma_\nu U_\mu^\dagger \Gamma_\nu^\dagger) U_\nu^\dagger \right\} + 2\beta N^2 \]

\[ = -N\beta \sum_{\mu \neq \nu} Z_{\nu\mu} \text{tr} \left( V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger \right) + 2\beta N^2 \]

where \( V_\mu \equiv U_\mu \Gamma_\mu \) are unitary matrices.

Nothing but Twisted Eguchi-Kawai model Gonzalez-Arroyo, Okawa '83; Eguchi, Nakayama '83.
**Ginsparg-Wilson Dirac operator** Nishimura, Vazquez-Mozo '01; Aoki, Iso, Nagao '02

\[ D_{GW} = \frac{1}{a} (1 - \Gamma \hat{\Gamma}) \]

where Neuberger '98

\[ \Gamma = \gamma_5 \]
\[ \hat{\Gamma} = \frac{H}{\sqrt{H^2}} \]
\[ H = \gamma_5 (1 - aD_W) \]

where \( D_W \) is the Wilson Dirac operator

\[ D_W = \frac{1}{2} \sum_{\mu=1}^{d} \left\{ \gamma_\mu (\nabla_\mu^* + \nabla_\mu) - a \nabla_\mu^* \nabla_\mu \right\} \]

\[ \nabla_\mu \Psi = \frac{1}{a} \left[ U_\mu (\Gamma_\mu \Psi \Gamma_\mu^\dagger) - \Psi \right] = \frac{1}{a} \left[ V_\mu \Psi \Gamma_\mu^\dagger - \Psi \right], \]

\[ \nabla_\mu^* \Psi = \frac{1}{a} \left[ \Psi - (\Gamma_\mu^\dagger U_\mu^\dagger \Gamma_\mu)(\Gamma_\mu^\dagger \Psi \Gamma_\mu) \right] = \frac{1}{a} \left[ \Psi - V_\mu^\dagger \Psi \Gamma_\mu \right] \]
**GW relation** is satisfied by the definition.

\[ \Gamma D_{GW} + D_{GW} \hat{\Gamma} = 0 \]

\[ (\Leftrightarrow \Gamma D_{GW} + D_{GW} \Gamma = aD_{GW} \Gamma D_{GW}) \]

**Index of the GW Dirac operator**

\[ \text{Index}(D_{GW}) \equiv n_+ - n_- = \frac{1}{2} \text{Tr}(\Gamma + \hat{\Gamma}) \]

where \( n_{\pm} \) is the number of zero-modes with \( \pm \) chirality.

\( \text{Tr} \) represents a trace over the space of matrix and the spinor index.

1. \( \frac{1}{2} \text{Tr}(\Gamma + \hat{\Gamma}) \) takes only integer values, since \( \Gamma \) and \( \hat{\Gamma} \) have the form of sign operator.

2. In the continuum limit, it takes the form of the topological charge. [Iso, Nagao '02]

3. For topologically nontrivial gauge backgrounds, it takes the corresponding nonzero values.
Topological charge on finite NCG

\[
Q = \frac{1}{4\pi i} \sum_x \sum_{\mu\nu} \epsilon_{\mu\nu} U_\mu(x) * U_\nu(x + a\hat{\mu}) * U_\mu(x + a\hat{\nu})^* * U_\nu(x)^* ,
\]

\[
= \frac{1}{4\pi i} N \sum_{\mu\nu} \epsilon_{\mu\nu} \mathcal{Z}_{\nu\mu} \text{tr} \left( V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger \right)
\]

A naive discretization of the one in the continuum theory. 
\(Q\) takes non-integer values in general.

cf) A more sophisticated definition are given in lattice theory.  
Hasenfratz '98
§3. Analysis with classical solutions

General classical solutions van Baal '83, Griguolo, Seminara '03

\[ V_{\mu} = \begin{pmatrix} \Gamma^{(1)}_{\mu} \\ \Gamma^{(2)}_{\mu} \\ \vdots \\ \Gamma^{(k)}_{\mu} \end{pmatrix} \]

where

\[ \Gamma^{(j)}_{\mu} \Gamma^{(j)}_{\nu} = Z^{(j)}_{\mu\nu} \Gamma^{(j)}_{\nu} \Gamma^{(j)}_{\mu} \]

\[ Z^{(j)}_{12} = Z^{(j)*}_{21} = \exp \left( 2\pi i \frac{m_j}{n_j} \right) \]
Index vs Topological charge

ν ≈ Q ⇒ index theorem

(Scatter plot for classical solutions with small action)

N = 25

S/β < 100

S/β < 200

S/β < 300

S/β < 600
Topological charge vs action

Scatter plot for the classical solutions (black blobs are of one-block-solutions)

\[ N = 25 \]

\[ N = 75 \]
• For the classical solutions

\[
S = 4N\beta \sum_{1 \leq j \leq k} n_j \sin^2 \left\{ \pi \left( \frac{m_j}{n_j} - \frac{M}{N} \right) \right\}
\]

\[
Q = \frac{N}{2\pi} \sum_{1 \leq j \leq k} n_j \sin \left\{ 2\pi \left( \frac{m_j}{n_j} - \frac{M}{N} \right) \right\}
\]

• In \( N \to \infty \), for finite action configurations,

\[
Q \simeq N \left( \sum m_j - M \right)
\]

\( Q \) takes only multiple integers of \( N \).
Minimum action is given by one-block-solution

\[
S \simeq 4\pi^2 \beta \left( \frac{Q}{N} \right)^2
\]

• In \( \beta \to \infty \), only \( Q = 0 \) sector survives.
Comparison to the Commutative case

- General classical solutions (up to moduli)

\[
U_1(x) = \begin{cases} 
1 & \text{if } x_1 \neq a(N - 1) \\
\exp(-2\pi i x_2 Q/aN) & \text{if } x_1 = a(N - 1)
\end{cases}
\]

\[
U_2(x) = \exp(2\pi i x_1 Q/aN^2)
\]

with topological charge \( Q \) (arbitrary integer values)

\[
S_{\text{classical}}(Q) \approx \frac{4\pi^2 \beta}{N^2} Q^2
\]

- When \( Q \) is a multiple integer of \( N \)

\( \Rightarrow \) Translationally invariant in 2nd direction \( \Rightarrow \) Star product becomes ordinary product

\( \Rightarrow \) The above configurations are the classical solutions in NC theory as well.

Giusti, Gonzalez-Arroyo, Hoelbling, Neuberger, Rebbi '02, Kiskis, Narayanan, Neuberger '02

- For the other cases, situations change since singularity is not allowed in the NCG.
§4. Analysis by Monte Carlo simulation

Model and Observable (again)

- **Model**

\[
S_{\text{matrix}} = -N\beta \sum_{\mu \neq \nu} Z_{\nu \mu} \text{tr} \left( V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger \right) + 2\beta N^2
\]

- **Observe**

\[
\nu \equiv \text{Index}(D_{GW}) = \frac{1}{2} Tr (\Gamma + \hat{\Gamma}) = \frac{1}{2} Tr \frac{H}{\sqrt{H^2}}
\]

where

\[
H = \gamma_5 (1 - aD_W)
\]

- **Obtain the probability distribution (or histogram) of the index \( \nu \).**
Probability distribution of index $P(\nu)$

- $\beta = 0$

ie, without taking account of the action

- Topologically nontrivial configurations with finite probability.
- Same distribution of $\nu/N$, for various $N$.
- $P(\nu)$ is asymmetric under $\nu \leftrightarrow -\nu$. \(\Rightarrow\) Generation of index.

We can expect it quite generically from the parity violation due to the NCG.
\* \( \beta \neq 0 \)

- \( \beta = 0 \)

- \( \beta = 0.1 \)

- \( \beta = 0.2 \)

- \( \beta = 0.3 \)

- \( \beta = 0.4 \)

- \( \beta = 0.5 \)

- \( \beta = 0.6 \)

- \( \beta = 0.7 \)

- \( P(\nu) \rightarrow \delta_{\nu 0} \) for increasing \( \beta \) and for increasing \( N \).

- Continuum limit: \( N, \beta \rightarrow \infty \) with \( \theta = N/\beta \) fixed. Bietenholz, Hofheinz, Nishimura '02

- The above behavior is consistent with the instanton calculus in the continuum theory. Paniak, Szabo '02
$P(\nu)/P(0)$ for $\nu = \pm 1$ versus $\beta$ and $N$

Exponentially decreasing behavior of $P(\nu)/P(0)$ as a function of $\beta$ and of $N$. 

various $\beta$ at $N = 15$

various $N$ at $\beta = 0.55$
Average action in each topological sector

various $\beta$ at $N = 15$

A dip at $\nu = 0$ and plateau for $\nu \neq 0$

various $N$ at $\beta = 0.55$
Depth of the dip

\[ \Delta S \equiv \bar{S}(\nu = -1) - \bar{S}(\nu = 0) \]

\[ \Delta S \text{ vs } \beta \text{ for } N = 15 \]

\[ \Delta S \text{ vs } N \text{ for various } \beta \]

Linear dependence \( \Rightarrow \)
Consistent with the exponentially decreasing behavior of \( P(\nu)/P(0) \).
Consistent with the analysis about the classical solutions.
• So far, we have considered in the trivial module.

• Topologically nontrivial sectors in the gauge theory on the discretized NC torus can be constructed by using the projective module in the twisted reduced model, usually labeled by two integers $(p, q)$. Griguolo, Seminara '03

\[ S = -N\beta \sum_{\mu \neq \nu} Z_{\nu\mu} \text{tr} \left( V_{\mu} V_{\nu} V_{\mu}^\dagger V_{\nu}^\dagger \right) + 2\beta N^2 \]

Matrix size $N$ and the twist $Z_{\mu\nu}$ are tuned by the integers $p, q$. For the small action configurations, the topological charge $Q$ takes the value

\[ Q = q \]
Preliminary results of Monte Carlo simulations
Probability distribution of the index $\nu$

$P(\nu) \rightarrow \delta_{\nu 1}$ for increasing $\beta$ and for increasing $N$.

We also observed a dip at $\nu = 1$ in the average action.
§5. Conclusion and discussion

- GW relation is useful for defining the index in NCG or MM.

- We found new dynamical aspects of topology peculiar to NCG.
  - Generation of nontrivial index
  - A single sector survives the continuum limit
    \[ \Rightarrow \text{chiral fermion in our 4-dimensional world in string theories} \]
    \[ \Rightarrow \text{baryogenesis, solution to strong CP problem} \]

- Extension to 4 and higher dimensions such as \( T^4, S^2 \times S^2, \text{CP}^2 \).

- Extension to the study of the topology of spacetime.

- Study about how to embed (curved) space and matter in MM.