Numerical Calabi-Yau metrics and hermitian Yang-Mills connections

Robert L. Karp
Rutgers University

Joint with: Mike Douglas, Sergio Lukic, Rene Reinbacher

Outline

1. Heterotic compactifications & Vector bundle moduli
2. CY metrics from FS metrics
3. “Experimental results”

References

hep-th/0606261 + 2 more soon

Two general approaches

- Spaces w/ lots of symmetry – T^n
- Explicit solutions
- Hard to make contact w/ SM

Two general approaches

- Spaces w/ lots of symmetry – T^n
- Explicit solutions
- Hard to make contact w/ SM

- Less/NO symmetry CY
- No explicit solutions
- More promising phenomenology
- Heterotic string compactifications on a CY

1. Vector bundle moduli

- Fields in LEEA
- N=1 SUSY
- W superpotential
- Yukawa couplings
- Quasi-topological "holomorphic"
- Can be determined w/o knowledge of g

1. Vector bundle moduli

- Fields in LEEA
- N=1 SUSY
- W superpotential
- Yukawa couplings
- Quasi-topological "holomorphic"
- Can be determined w/o knowledge of g

- Kinetic term
- Canonically normalize fields
- Kahler potential
- Non-holomorphic
- Need Kahler metric on moduli space of vector bundles

“Our” approach

(1) Find the Ricci flat metric
Yau's thm: fix Kahler class, cpx str $\quad \exists!\, g$

(2) Find hYM - Donaldson-Uhlenbeck-Yau
stable bdls $\Rightarrow$ hYM solutions

(3) Bundle deformations $\Rightarrow$ def of hYM solutions

(4) Compute the Kahler metric on moduli space of hYM solutions

2. How do we find Ricci flat metrics?

- Use FS metrics, embeddings and restrict
- Tian, Yau, Donaldson, ...
- Use the global sections of an ample line bdle

2. How do we find Ricci flat metrics?

- Use FS metrics, embeddings and restrict
- Tian, Yau, Donaldson, ...
- Use the global sections of an ample line bdle
- Ex. E = ell. curve.; cubic in P^2 , L = $O_E(3)$

(x_0, x_1, x_2) homogenous coords

$$x_0^3,\ x_1^3,\ x_2^3,\ x_0^2x_1,\ x_0^2x_2,\ x_0x_1^2,\ x_0x_2^2,\ x_1x_2^2,\ x_1^2x_2,\ x_0x_1x_2$$

- No common zero
- 10 sections -1 = 8 dim proj space

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_E \longrightarrow 0$$

More generally:

X a Kahler mf, CY, L ample line bdle

use $H^0(X, L^{\otimes k}) \cong \mathbb{C}^{N_k+1}$ and get $i_k : X \hookrightarrow \mathbb{P}^{N_k}$

$$i_k(x_0, \ldots, x_n) = (s_0(x), s_2(x), \ldots, s_N(x))$$

More generally:

X a Kahler mf, CY, L ample line bdle

$$\text{use } H^0(X, L^{\otimes k}) \cong \mathbb{C}^{N_k+1} \text{ and get } i_k : X \hookrightarrow \mathbb{P}^{N_k}$$

$$i_k(x_0, \ldots, x_n) = (s_0(x), s_2(x), \ldots, s_N(x))$$

- Given the embedding what's next?
- Try to find a good FS metric on P and restrict it to X

More generally:

X a Kahler mf, CY, L ample line bdle

$$\text{use } H^0(X, L^{\otimes k}) \cong \mathbb{C}^{N_k+1} \text{ and get } i_k : X \hookrightarrow \mathbb{P}^{N_k}$$

$$i_k(x_0, \ldots, x_n) = (s_0(x), s_2(x), \ldots, s_N(x))$$

- Given the embedding what's next?
- Try to find a good FS metric on P and restrict it to X
- Why would it approximate a CY metric?
- Take the Kahler potential of FS on P^n $\quad k \log \sum_{i=0}^{n} |x_i|^2$
- Do some "algebra"

$$k \log \sum_{i=0}^{n} |x_i|^2 = \log \left(\sum_{i=0}^{n} |x_i|^2 \right)^k$$

$$= \log \sum_{i_1, \cdots, i_k = 0}^{n} x_{i_1} \cdots x_{i_k} \bar{x}_{i_1} \cdots \bar{x}_{i_k}$$

Generalizing this

$$\log \sum_{i_1, \cdots, i_k, j_1, \cdots, j_k = 0}^{n} h^{i_1 \cdots i_k j_1 \cdots j_k} x_{i_1} \cdots x_{i_k} \bar{x}_{j_1} \cdots \bar{x}_{j_k}$$

Observe:
- just like products of sections
- Can approximate large class by choice of h

Donaldson's theorems

- There is a notion to be balanced for an embedding $i_k : X \hookrightarrow \mathbb{P}^{N_k}$ given by (X, L^k) w/ choice of FS metric $$\log \sum_{i,j=0}^{N_k} h^{i,j} x_i \bar{x}_j$$
- Instead of def I'll give a procedure to find such pairs

Donaldson's theorems

- There is a notion to be balanced for an embedding $i_k : X \hookrightarrow \mathbb{P}^{N_k}$ given by (X, L^k) w/ choice of FS metric $\log \sum_{i,j=0}^{N_k} h^{i,j} x_i \bar{x}_j$
- Instead of def I'll give a procedure to find such pairs
- Theorem

$$\omega_k = \frac{1}{k} i_k^*(\omega_{FS(h)}) \longrightarrow \omega_{CY} = c_1(L) \text{ as } k \to \infty, \text{ error: } \frac{1}{k^2}$$

The T-map

- Donaldson

$$T(h)_{ij} = \frac{N_k + 1}{vol(X)} \int_X \frac{s_i \bar{s_j}}{\sum h^{ab} s_a \bar{s_b}} \, dvol_X$$

- For any h_0 hermitian matrix, $T^r(h_0)$ converges to the balanced metric.
- Therefore to find the CY metric we need
  - $r \longrightarrow \infty$, i.e., iterate T-map (10-15 steps w/ $10^{-5,-7}$)
  - Take larger and larger k's

“Experimental” setup

CY varieties $\sum_{i=0}^{n} x_i^{n+1} - (n+1)\psi \prod_{i=0}^{n} x_i = 0 \;\; \text{in} \;\; \mathbb{P}^n$

ψ = Cpx. Structure parameter

$\psi = 0$ Fermat point

- Elliptic curve: $x_0^3 + x_1^3 + x_2^3 - 3\psi x_0 x_1 x_2 = 0$

CY varieties $\sum_{i=0}^{n} x_i^{n+1} - (n+1)\psi \prod_{i=0}^{n} x_i = 0$ in $\mathbb{P}^n$

ψ = Cpx. Structure parameter

$\psi = 0$ Fermat point

- Elliptic curve: $x_0^3 + x_1^3 + x_2^3 - 3\psi x_0 x_1 x_2 = 0$
- K3 $x_0^4 + x_1^4 + x_2^4 + x_3^4 - 4\psi x_0 x_1 x_2 x_3 = 0$

CY varieties $\sum_{i=0}^{n} x_i^{n+1} - (n+1)\psi \prod_{i=0}^{n} x_i = 0 \;\text{ in }\; \mathbb{P}^n$

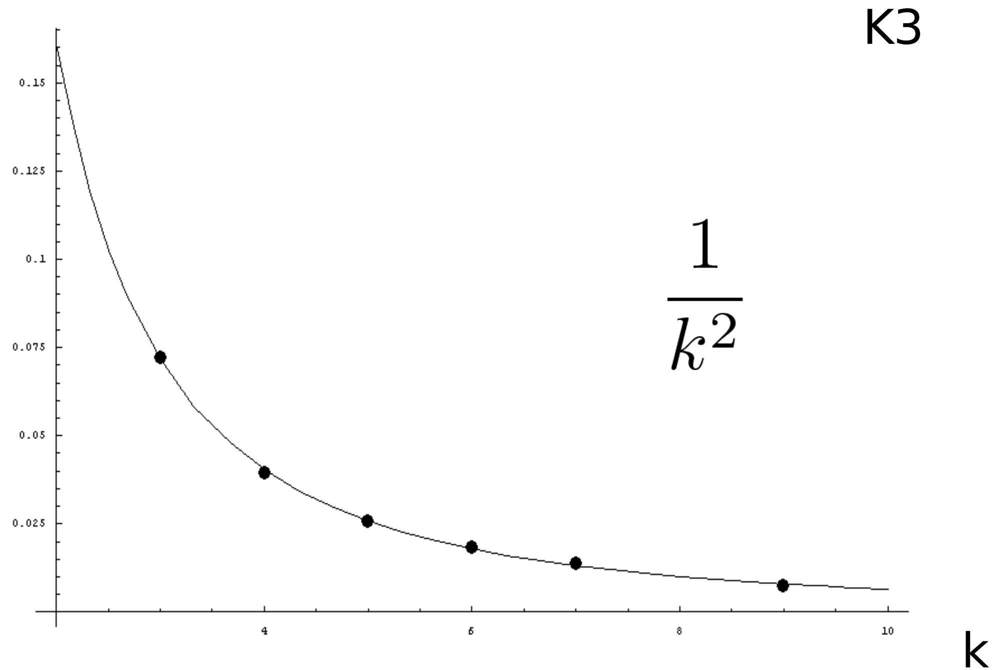
ψ = Cpx. Structure parameter

$\psi = 0$ Fermat point

- Elliptic curve: $x_0^3 + x_1^3 + x_2^3 - 3\psi x_0 x_1 x_2 = 0$
- K3 $x_0^4 + x_1^4 + x_2^4 + x_3^4 - 4\psi x_0 x_1 x_2 x_3 = 0$
- 3-fold $x_0^5 + \cdots + x_4^5 - 5\psi x_0 x_1 \cdots x_4 = 0$

The difference for a random point of

$$\omega_k = \frac{1}{k} i_k^*(\omega_{FS(k)}) - \omega_{CY} = c_1(L)$$

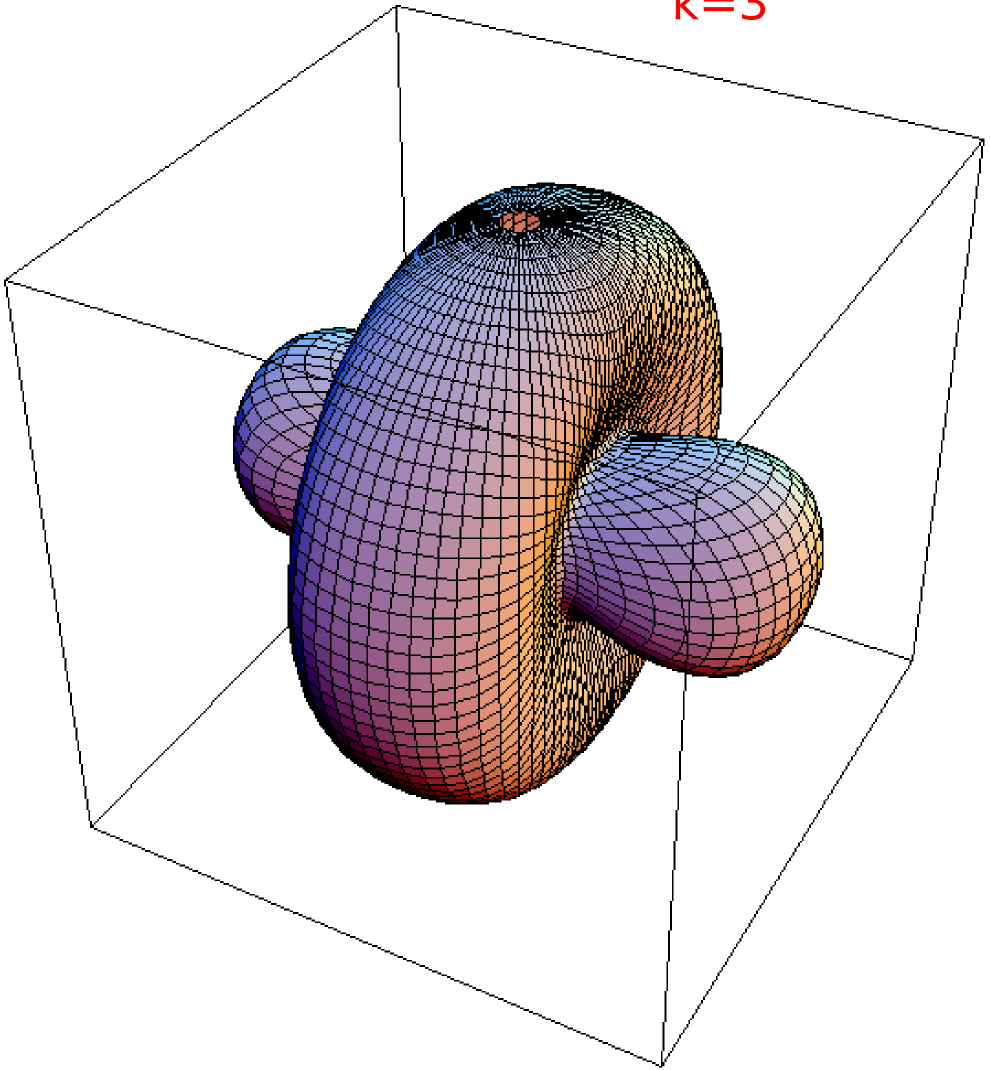


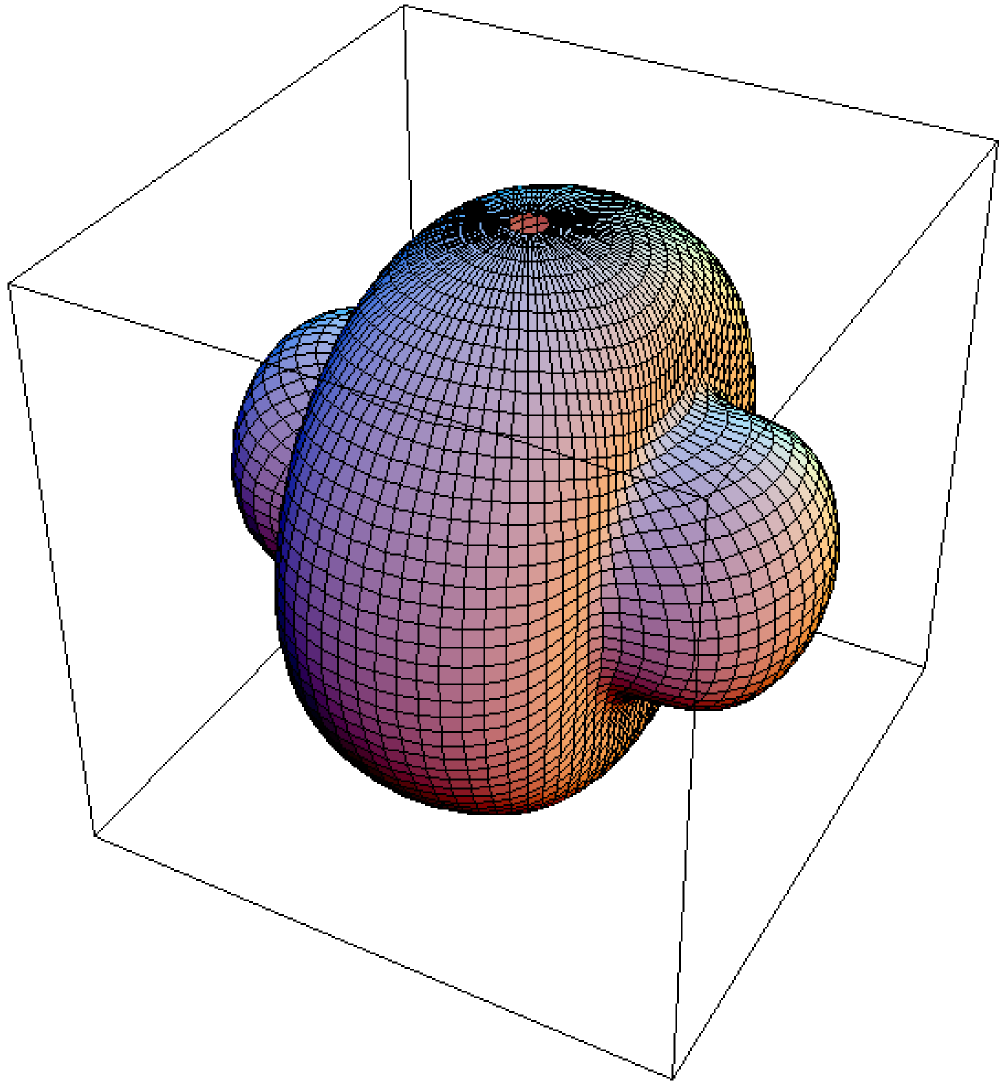
Same for more points

- Rational curve (1,-1,t,0,-t) which lies in the Quintic
- on the radial direction plot pointwise

$$\frac{\omega_k^3}{\Omega \wedge \bar{\Omega}}$$

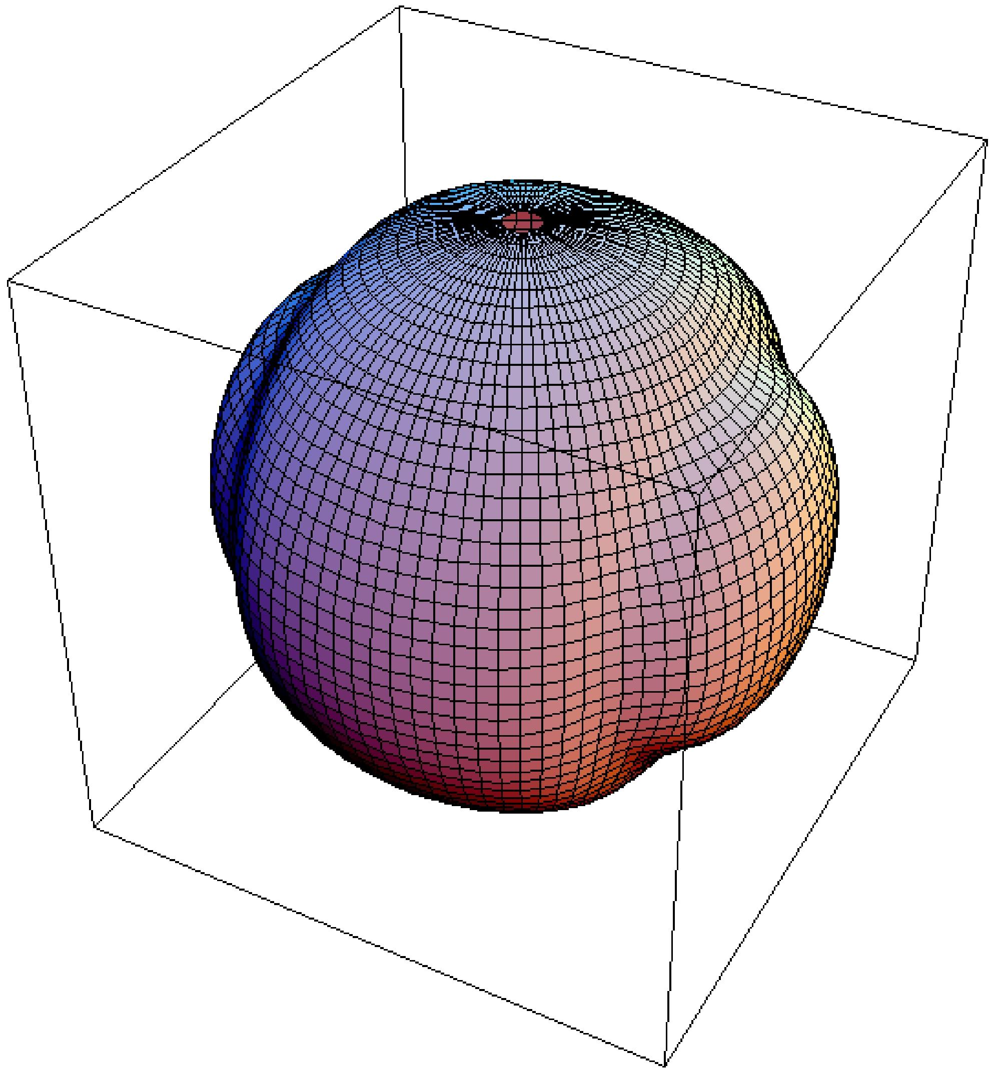
- WMAP
- Ideally 1

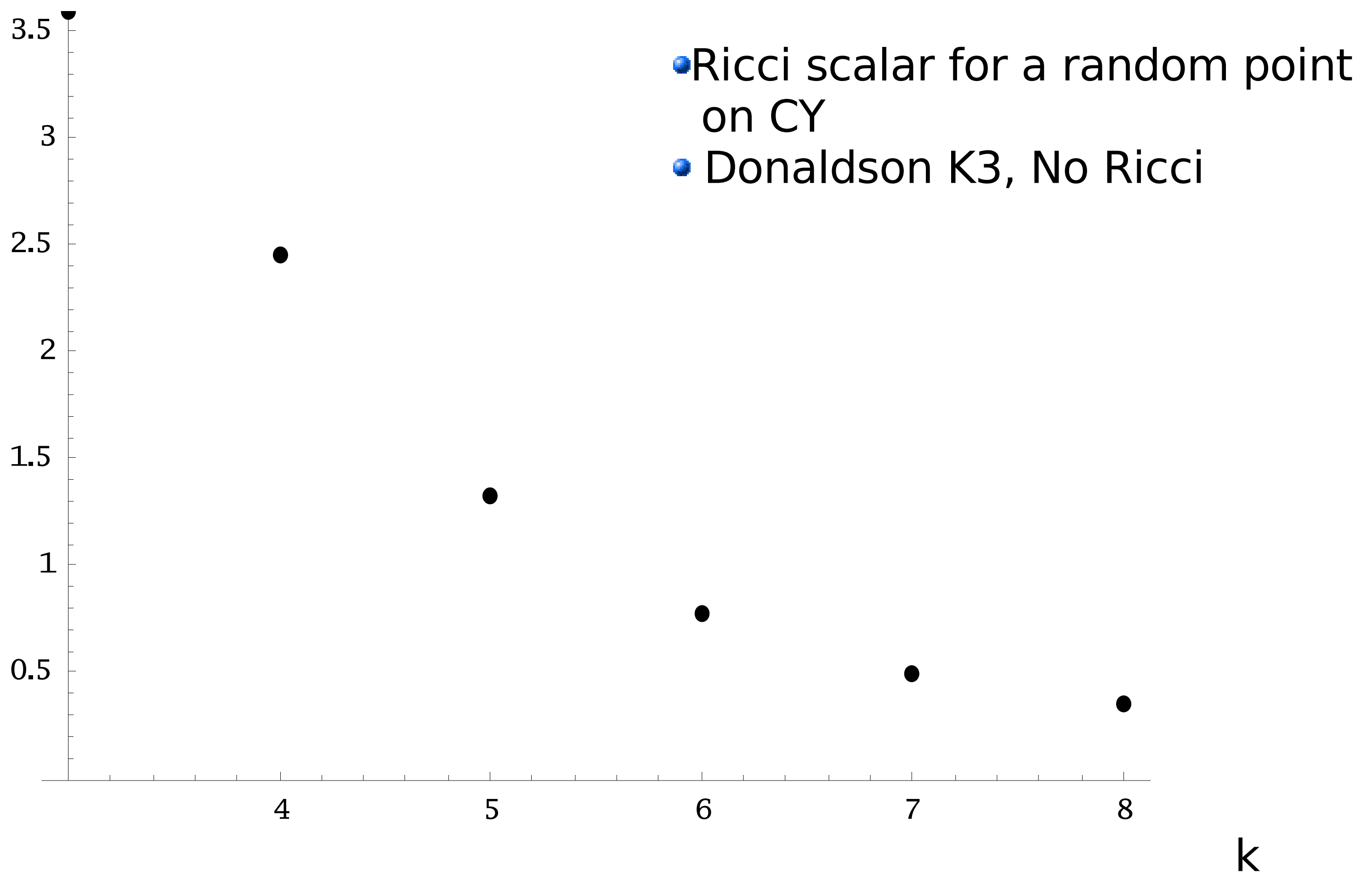




k=6

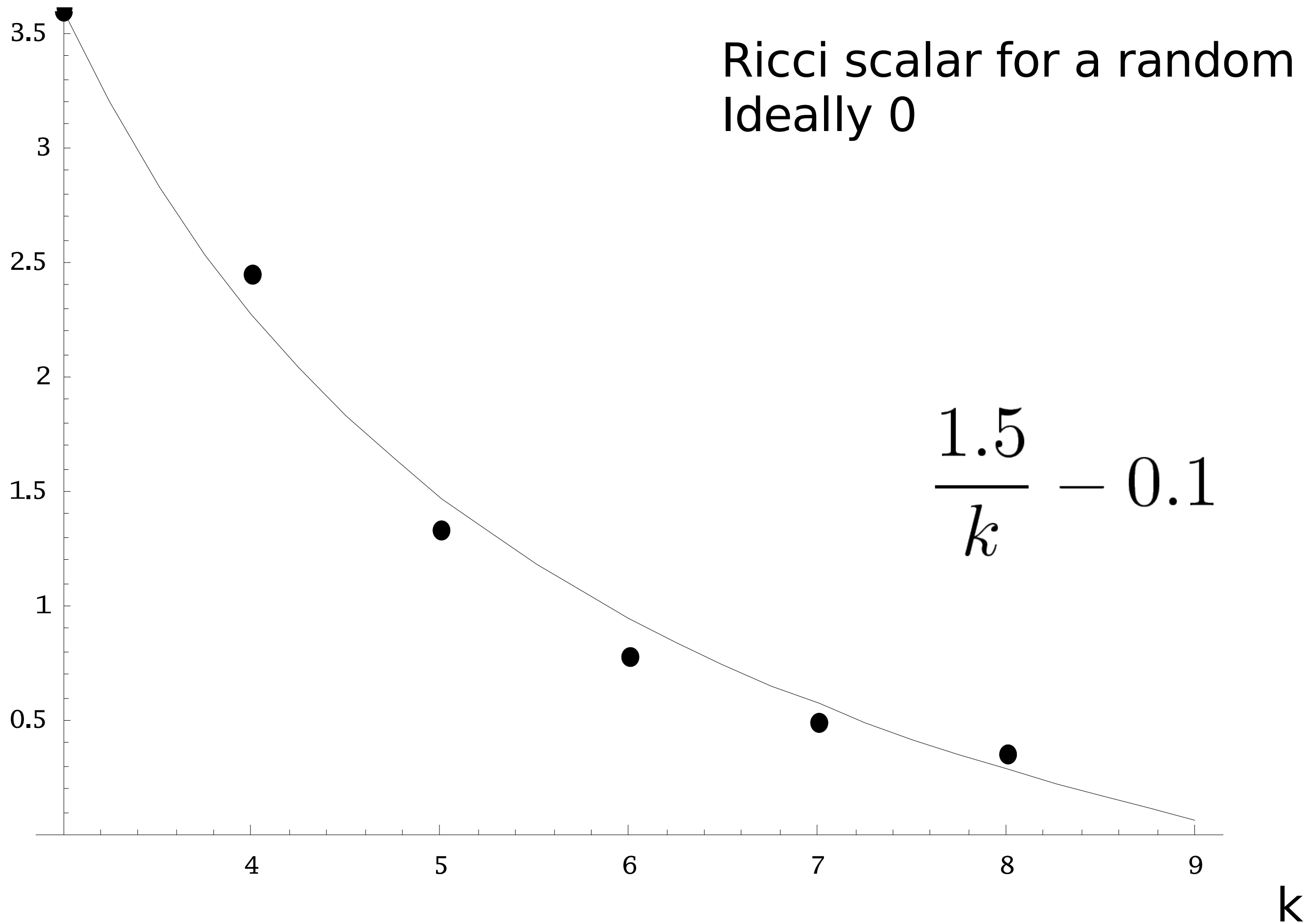
k=9



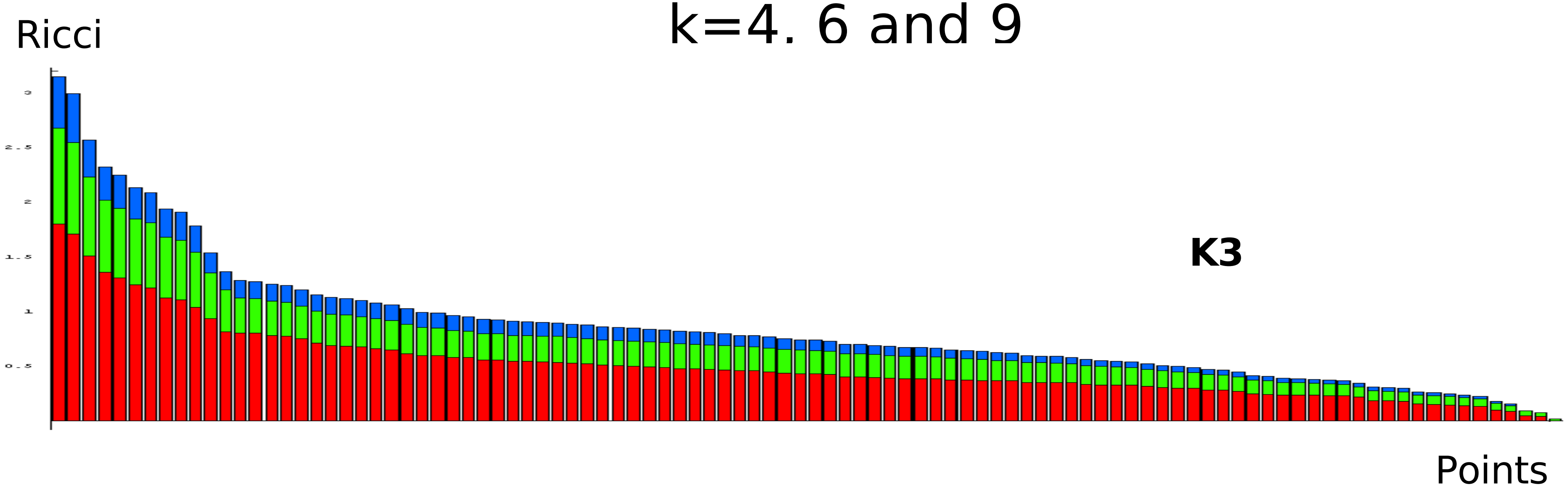

Ricci scalar for a random point on CY
Donaldson K3, No Ricci
3.5
3
2.5
2
1.5
1
0.5
4
5
6
7
8
k

Ricci scalar for a random point
Ideally 0

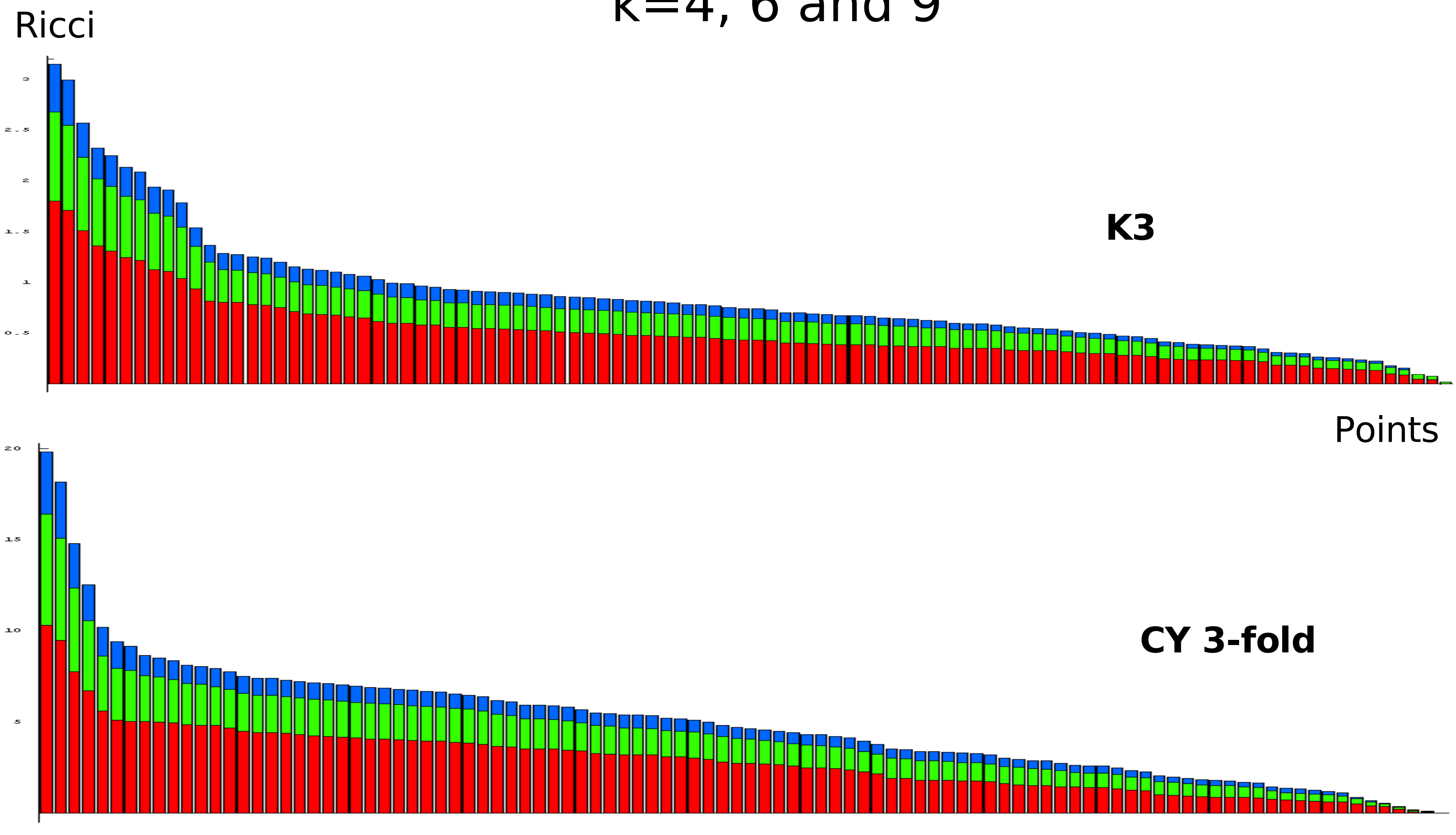
$$\frac{1.5}{k} - 0.1$$



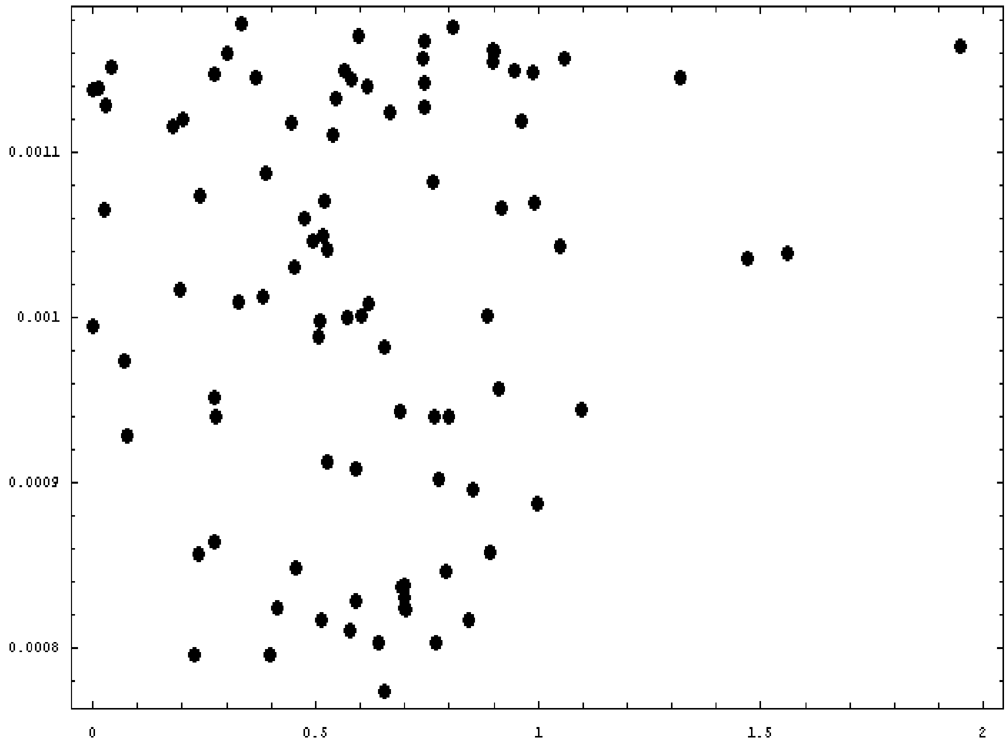
Distribution of Ricci scalars for 100 random points k=4. 6 and 9



Distribution of Ricci scalars for 100 random points k=4, 6 and 9



Scattered plot of points and masses



Numerics

- C++
- Ublas on par w/ Fortran 77
- Athlon 64 x2 4800+
- 4GB
- Cheaper than my laptop
- Hours not days

Conclusions

- More efficient than solving Monge-Ampere (was never done for CY, memory)
- T-map generalizes to hYM - Rene
- T-map can be improved (Donaldson)
- Numerics unavoidable? He atom; 3-body