Dirac-Kähler Twisted $\mathcal{N} = 4$ Super Yang-Mills Theory with Central Charge in Four Dimensions

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Joint Meeting of Pacific Region Particle Physics Communities in Hawaii
October 31, 2006
Dirac-Kähler Twist

Dirac-Kähler twist: a kind of topological twist

- applied for $\mathcal{N} = \#$(spinor components) $= 2^{D/2}$ SUSY in $D$-dimensional Euclidean space.

\[
D = 2 \rightarrow \mathcal{N} = 2, \quad D = 4 \rightarrow \mathcal{N} = 4, \ldots
\]

- defines tensor charges (twisted supercharges) of all rank,

\[
Q_{i\alpha}^{\text{D-K twist}} \rightarrow Q, \ Q_\mu, \ Q_{\mu\nu}, \ldots
\]

The scalar charge $Q$ can be interpreted as a BRST charge in a quantized TQFT.

- crucial to formulate lattice SUSY with

\[
\#$(supercharges) = \#$(doublers) $= 2^D,
\]

i.e.

\[
\mathcal{N} = 2^D / 2^{D/2} = 2^{D/2}.
\]

[D’Adda-Kanamori-Kawamoto-Nagata, Catterall et. al., Kaplan et. al., Sugino, ...]
Twisted lattice SUSY formalism

- $\mathcal{N} = D = 2 \rightarrow$ OK.
- $\mathcal{N} = D = 4 \rightarrow$ not yet...,
  due to the lack of simple superspace formulation of $\mathcal{N} = D = 4$ SUSY in continuum.

Motivation

In order to construct lattice SUSY formalism, we need Dirac-Kähler twisted superspace formalism as its continuum counterpart, especially for $\mathcal{N} = D = 4$. 
Central Charge Extension of $\mathcal{N} = D = 4$ SYM

Summary

Dirac-Kähler Twist for $\mathcal{N} = D = 4$

$\mathcal{N} = D = 4$ Dirac-Kähler Twisted SUSY

$\mathcal{N} = D = 4$ Dirac-Kähler TSYM (on-shell) has been analyzed so far. [Marcus, Kato-Kawamoto-Miyake, Blau-Thompson]

- The untwisted $\mathcal{N} = D = 4$ SUSY has two $SO(4)$ symmetries:

  $$ [J_{\mu \nu}, Q_{i\alpha}] = \frac{1}{2} (\gamma_{\mu \nu})_{\alpha \beta} Q_{i\beta}, \quad [R_{ab}, Q_{i\alpha}] = \frac{1}{2} (\gamma_{ab})_{ij} Q_{j\beta}. $$

- Define a new isometry:

  $$ \delta'_\text{rot} := \delta_{\text{Lorentz}} + \delta_{\text{R-symmetry}}, \quad J'_{\mu \nu} := J_{\mu \nu} \otimes 1 + 1 \otimes R_{\mu \nu}. $$

- Take the Clebsch-Gordan decomposition

  $$ Q_{i\alpha} = \sum_{p} \frac{1}{p!} Q_{\mu_1 \cdots \mu_p} (\gamma^{\mu_1 \cdots \mu_p})_{i\alpha}, $$

under which the supercharges transform as ”Lorentz” tensors

  $$ \delta'_\text{rot} Q_{\mu_1 \cdots \mu_p} = \sum_{i} \epsilon_{\mu_i [\mu} Q_{\mu_1 \cdots \mu] \cdots \mu_p}. $$
Dirac-Kähler Twisted SUSY with Central Charge

This work

Yet another possible $\mathcal{N} = D = 4$ Dirac-Kähler twisted SYM: the central charge extension. [Sohnius-Stelle-West]

- The untwisted $\mathcal{N} = D = 4$ SUSY w/ one central charge has the isometries

$$SO(4) \times USp(4) \cong SO(4) \times SO(5).$$

- $USp(4) \cong SO(5)$ (spinor repr.)

- $SO(4)$ and $SO(5)$ are represented by the same set of gamma matrices
  → we can carry out the D-K twisting as in the previous case.

- cf. [Yamron, Vafa-Witten]
Formulation

- We take the gauge-supercovariant derivative formulation:
  \[ D_A : \text{supercovariant derivative} \]
  \[ \rightarrow \nabla_A := D_A - i\Gamma_A : \text{gauge-supercovariant derivative.} \]

- Superfields are defined as the supercurvatures
  \[ \{\nabla_A, \nabla_B\} = -iF_{AB}, \]
  to be constrained to make them irreducible.

- The constraints are found as follows:
  \[ \mathcal{N} = 1 \ 10\text{D SYM} \quad \xrightarrow{\text{dimensional reduction}} \quad \mathcal{N} = 4 \ 5\text{D SYM} \]
  \[ \xrightarrow{\text{Legendre transformation}} \quad \mathcal{N} = 4 \ 4\text{D SYM w/ CC} \]
  \[ \xrightarrow{\text{D-K twist}} \quad \mathcal{N} = 4 \ 4\text{D TSYM w/ CC.} \]
Dimensional Reduction through Legendre Transformation

- From 5D to 4D, we carry out a Legendre transformation to obtain a \textit{gauged central charge}. [Sohnius-Stelle-West]

\[
(\Phi, \partial_5 \Phi) \rightarrow (\Phi, \Pi_\Phi), \quad \mathcal{L}(5) \rightarrow \mathcal{L},
\]

\[
\mathcal{L} := \mathcal{L}(5) - \Pi_\Phi \partial_5 \Phi, \quad \Pi_\Phi := \frac{\partial \mathcal{L}}{\partial (\partial_5 \Phi)} \quad \text{(conjugate to } \Phi),
\]

so that

\[
\begin{cases}
    \partial_5 \mathcal{L} = 0, & \text{no dependence on } x^5, \\
    (\Phi, \Pi_\Phi), & \text{independent fields,} \\
    P_5 \sim Z, & \text{central charge in 4D.}
\end{cases}
\]

These are on-shell in 5D, but \textit{off-shell} in 4D:

\[
\#(\text{fermions}) = 16,
\]

\[
\#(\text{bosons}) = 2(10 \ (10D \text{ gauge bosons}) - 1 \ (\text{gauge}) - 1 \ (\text{Lagrange multiplier})) = 16.
\]
We obtain the following constraints for the supercurvatures:

<table>
<thead>
<tr>
<th>$\mathcal{N} = D = 4$ TSYM w/ CC constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>${\nabla, \nabla} = {\tilde{\nabla}, \tilde{\nabla}} = 0$,</td>
</tr>
<tr>
<td>${\nabla_{\mu\nu}, \nabla_{\rho\sigma}} = 2i\varepsilon_{\mu\nu\rho\sigma}(\nabla_5 - W)$,</td>
</tr>
<tr>
<td>${\nabla_{\mu}, \nabla_{\nu}} = {\tilde{\nabla}<em>{\mu}, \tilde{\nabla}</em>{\nu}} = 0$,</td>
</tr>
<tr>
<td>${\nabla, \nabla_{\mu\nu}} = {\tilde{\nabla}, \tilde{\nabla}_{\mu\nu}} = 0$,</td>
</tr>
<tr>
<td>${\nabla, \nabla_{\mu}} = 2i(\nabla_{\mu} - W_{\mu})$,</td>
</tr>
<tr>
<td>${\nabla, \tilde{\nabla}<em>{\mu}} = {\tilde{\nabla}, \nabla</em>{\mu}} = 0$,</td>
</tr>
<tr>
<td>${\nabla_{\mu\nu}, \nabla_{\rho}} = 2i\delta_{\mu\nu, \rho}^{\sigma}(\nabla_{\sigma} + W_{\sigma})$,</td>
</tr>
<tr>
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<tr>
<td>${\nabla_{\mu\nu}, \nabla} = 2i(\nabla_{\mu} - W_{\mu})$,</td>
</tr>
<tr>
<td>${\tilde{\nabla}<em>{\mu\nu}, \nabla</em>{\rho}} = 2i\delta_{\mu\nu, \rho}^{\sigma}(\nabla_{\sigma} - W_{\sigma})$,</td>
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<td>${\nabla_{\mu\nu}, \tilde{\nabla}} = 2i\varepsilon_{\mu\nu, \rho}^{\sigma}(\nabla_{\sigma} + W_{\sigma})$.</td>
</tr>
</tbody>
</table>
Solving the constraints, we find the following independent fields:

**Solutions—Independent Component Fields**

\[
\mathcal{F}_{\mu\nu} := [\nabla_{\mu}, W_{\nu}], \quad \mathcal{F}_{\mu} := [\nabla_{\mu}, W],
\]
\[
H_{\mu} := [\nabla_{5}, W_{\mu}], \quad H := [\nabla_{5}, W], \quad g_{\mu} := [\nabla_{\mu}, \nabla_{5}],
\]
\[
\phi := W|, \quad \phi_{\mu} := W_{\mu}|, \quad (5 \text{ scalars})
\]
\[
\rho := \frac{1}{4} \mathcal{F}_{\mu,\mu}|, \quad \tilde{\rho} := \frac{1}{4} \tilde{\mathcal{F}}_{\mu,\mu}|, \quad \chi_{\mu\nu} := \frac{1}{2} \mathcal{F}_{\mu,\nu}|,
\]
\[
\lambda_{\mu} := \mathcal{F}_{\mu}|, \quad \tilde{\lambda}_{\mu} := \tilde{\mathcal{F}}_{\mu}|,
\]
\[
A_{\mu} := i \nabla_{\mu}|, \quad (1 \text{ gauge vector})
\]
\[
H := \frac{i}{2} H|, \quad H_{\mu} := \frac{i}{2} H_{\mu}|, \quad (5 \text{ auxiliary fields})
\]
\[
V_{\mu} := \frac{i}{2} g_{\mu}|. \quad (1 \text{ pseudo vector, constrained})
\]
Supertransformations are also given by solving the constraints. In general component filed $f = F|$, we obtain $Q_A f := [\nabla_A, F]|$.

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>$Q \phi = -i \rho$, $\tilde{Q} \phi = -i \rho$,</td>
</tr>
<tr>
<td>$Q \phi_\mu = -i \lambda_\mu$, $\tilde{Q} \phi_\mu = +i \tilde{\lambda}_\mu$,</td>
</tr>
<tr>
<td>$Q \rho = 2H - [\nabla_\mu, \phi^\mu]$, $\tilde{Q} \rho = 0$,</td>
</tr>
<tr>
<td>$Q \tilde{\rho} = 0$, $\tilde{Q} \tilde{\rho} = 2H + [\nabla_\mu, \phi^\mu]$,</td>
</tr>
<tr>
<td>$Q \chi_{\mu\nu} = iF_{\mu\nu} - [\phi_\mu, \phi_\nu] + [\nabla_{[\mu}, \phi_{\nu]}$,</td>
</tr>
<tr>
<td>$\tilde{Q} \chi_{\mu\nu} = i\tilde{F}<em>{\mu\nu} - \frac{1}{2} \varepsilon</em>{\mu\nu\rho\sigma}[\phi_\rho, \phi_\sigma] - \varepsilon_{\mu\nu}^{\rho\sigma} [\nabla_\rho, \phi_\sigma]$,</td>
</tr>
<tr>
<td>$Q \lambda_\mu = 0$, $\tilde{Q} \lambda_\mu = -2(V_\mu - H_\mu)$,</td>
</tr>
<tr>
<td>$Q \tilde{\lambda}<em>\mu = -2(V</em>\mu + H_\mu)$, $\tilde{Q} \tilde{\lambda}_\mu = 0$,</td>
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</tbody>
</table>
Supertransformations

Solutions—Supertransformations (cont’d)

\[ QA_\mu = \lambda_\mu, \quad \tilde{Q}A_\mu = \tilde{\lambda}_\mu, \]
\[ QH = -\frac{i}{2} \left( [\nabla_\mu, \lambda^\mu] + [\lambda_\mu, \phi^\mu] \right), \]
\[ \tilde{Q}H = -\frac{i}{2} \left( [\nabla_\mu, \tilde{\lambda}^\mu] - [\tilde{\lambda}_\mu, \phi^\mu] \right), \]
\[ QH_\mu = \frac{i}{2} \left( [\nabla_\mu, \tilde{\rho}] - [\nabla_\rho, \tilde{\chi}^\rho_\mu] - [\tilde{\rho}, \phi_\mu] + [\tilde{\chi}^\rho_\mu, \phi_\rho] \right) + i[\lambda_\mu, \phi], \]
\[ \tilde{Q}H_\mu = -\frac{i}{2} \left( [\nabla_\mu, \rho] - [\nabla_\rho, \chi^\rho_\mu] + [\rho, \phi_\mu] - [\chi^\rho_\mu, \phi_\rho] \right) + i[\tilde{\lambda}_\mu, \phi], \]
\[ QV_\mu = -QH_\mu, \quad \tilde{Q}V_\mu = +\tilde{Q}H_\mu, \]

and for the other charges \((Q_\mu, \tilde{Q}_\mu, Q_{\mu\nu}).\)
Central Charge Extension of $\mathcal{N} = D = 4$ SYM

Solutions

Action

Solutions—Off-shell Action

$$
\mathcal{L} = \text{Tr} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} V_{\mu} V^\mu + \frac{1}{2} H_{\mu} H^\mu + \frac{1}{2} H H \\
- \frac{1}{4} [\nabla_\mu, \phi] [\nabla_\mu, \phi] - \frac{1}{4} [\nabla_\mu, \phi_\nu] [\nabla_\mu, \phi^\nu] \\
+ \frac{i}{2} \rho [\nabla_\mu, \lambda^\mu] + \frac{i}{2} \tilde{\rho} [\nabla_\mu, \tilde{\lambda}^\mu] + \frac{i}{2} \chi_{\mu\nu} [\nabla_\nu, \lambda_\mu] + \frac{i}{2} \tilde{\chi}_{\mu\nu} [\nabla_\nu, \tilde{\lambda}_\mu] \\
- i \rho [\phi_\mu, \tilde{\lambda}^\mu] - i \tilde{\rho} [\phi_\mu, \lambda^\mu] + i \chi^{\mu\nu} [\phi_\nu, \tilde{\lambda}_\mu] + i \tilde{\chi}^{\mu\nu} [\phi_\nu, \lambda_\mu] \\
- \frac{i}{2} \lambda_\mu [\phi, \lambda^\mu] + \frac{i}{2} \tilde{\lambda}_\mu [\phi, \tilde{\lambda}^\mu] - \frac{i}{2} \rho [\phi, \rho] + \frac{i}{2} \tilde{\rho} [\phi, \tilde{\rho}] \\
- \frac{i}{8} \chi^{\mu\nu} [\phi, \chi_{\mu\nu}] + \frac{i}{8} \tilde{\chi}^{\mu\nu} [\phi, \tilde{\chi}_{\mu\nu}] \\
+ \frac{1}{4} [\phi, \phi_\mu] [\phi, \phi^\mu] + \frac{1}{4} [\phi_\mu, \phi_\nu] [\phi^\mu, \phi^\nu] \right]
$$
The action is manifestly invariant under $Q$ and $\tilde{Q}$ as it is $Q$ and $\tilde{Q}$-exact:

$$\mathcal{L} = Q\mathcal{L} = \tilde{Q}\tilde{\mathcal{L}}.$$ 

The other invariances are also shown manifestly, since

$$Q_{(\mu} L_{\nu)} = \frac{1}{2} \delta_{\mu\nu} \mathcal{L} = \tilde{Q}_{(\mu} \tilde{L}_{\nu)},$$

$$Q_{\mu\nu} L_{\rho\sigma} = \frac{1}{12} \delta_{\mu\nu,\rho\sigma} \mathcal{L}$$

$$+ Q_{\mu\rho} L_{\nu\sigma} - Q_{\mu\sigma} L_{\nu\rho} - Q_{\nu\rho} L_{\mu\sigma} + Q_{\nu\sigma} L_{\mu\rho},$$

where

$$Q^\mu L_\mu = \frac{1}{2} Q^{\mu\nu} L_{\mu\nu} = \mathcal{L},$$

which can be rewritten as

$$\mathcal{L} \sim Q_\mu L_\mu \sim \tilde{Q}_\mu \tilde{L}_\mu \sim Q_{\mu\nu} L_{\mu\nu} \text{ no summation.}$$
### Manifest Invariance

#### Explicit Formulas

\[ L = \frac{1}{4} \left( \rho \tilde{Q} \tilde{\rho} - \tilde{\lambda}_\rho \tilde{Q} \chi_\rho + \frac{1}{2} \tilde{\chi}_\rho \sigma \tilde{Q} \chi^{\rho \sigma} \right), \]

\[ L_\mu = \frac{1}{16} \left( -\tilde{\rho} \tilde{Q}_\mu \rho + \tilde{\lambda}_\rho \tilde{Q}_\mu \lambda^\rho + \lambda_\rho \tilde{Q}_\mu \tilde{\lambda}^\rho - \frac{1}{2} \tilde{\chi}_\rho \sigma \tilde{Q}_\mu \chi^{\rho \sigma} \right), \]

\[ L_{\mu \nu} = \frac{1}{24} \left( \tilde{\rho} \tilde{Q}_{\mu \nu} \rho + \rho \tilde{Q}_{\mu \nu} \tilde{\rho} - \tilde{\lambda}_\rho \tilde{Q}_{\mu \nu} \lambda^\rho - \lambda_\rho \tilde{Q}_{\mu \nu} \tilde{\lambda}^\rho + \frac{1}{2} \tilde{\chi}_\rho \sigma \tilde{Q}_{\mu \nu} \chi^{\rho \sigma} \right), \]

\[ \tilde{L} = \frac{1}{4} \left( \tilde{\rho} Q_\rho - \lambda_\rho Q \tilde{\chi}^\rho + \frac{1}{2} \tilde{\chi}_\rho \sigma Q \chi^{\rho \sigma} \right), \]

\[ \tilde{L}_\mu = \frac{1}{16} \left( -\rho Q_\mu \tilde{\rho} + \tilde{\lambda}_\rho Q_\mu \lambda^\rho + \lambda_\rho Q_\mu \tilde{\lambda}^\rho - \frac{1}{2} \tilde{\chi}_\rho \sigma Q_\mu \chi^{\rho \sigma} \right), \]

\[ \tilde{L}_{\mu \nu} = \frac{1}{24} \left( \tilde{\rho} Q_{\mu \nu} \rho + \rho Q_{\mu \nu} \tilde{\rho} - \tilde{\lambda}_\rho Q_{\mu \nu} \lambda^\rho - \lambda_\rho Q_{\mu \nu} \tilde{\lambda}^\rho + \frac{1}{2} \tilde{\chi}_\rho \sigma Q_{\mu \nu} \chi^{\rho \sigma} \right). \]
The SUSY algebra closes off-shell by construction.

But the off-shell multiplet contains a constraint field $V_\mu$, with the constraint

$$\left[ \nabla_\mu, V_\mu \right] + \left[ W, H \right] + \left[ W_\mu, H_\mu \right] + \{ \rho, \tilde{\rho} \} + \{ \lambda_\mu, \tilde{\lambda}_\mu \} + \frac{1}{2} \{ \chi_{\mu\nu}, \tilde{\chi}_{\mu\nu} \} = 0.$$

The solution of the constraint: nonlocal. [Siegel]

Lagrange multiplier technique to impose the constraint $\rightarrow$ on-shell $\mathcal{N} = D = 4$ standard SYM multiplet.

Topological aspects: all the twisted supercharges $Q_A$ are nilpotent $\rightarrow$ BRST-like charges, with ghost number essentially comes from the chirality. Especially,

$$T_{\mu\nu} = \{ Q, \Lambda_{\mu\nu} \} = \{ \tilde{Q}, \tilde{\Lambda}_{\mu\nu} \} : \text{energy-momentum tensor.}$$
In this talk, the exact off-shell formulation of $N = D = 4$ Dirac-Kähler-twisted SYM theory with a central charge is manifestly presented on the twisted superspace.

- The invariance of the presented action under all of the sixteen twisted supercharges can be shown explicitly, as well as the fact that the twisted SUSY algebra closes off-shell.
- The theory should be interpreted as a topological field theory as in the other twisted supersymmetric theories.

- The resultant theory could play an interesting role from, especially, the viewpoint of a lattice SUSY formulation.