Dirac-Kähler Twisted $\mathcal{N} = 4$ Super Yang-Mills Theory with Central Charge in Four Dimensions

Jun Saito
in collaboration with: Noboru Kawamoto

Department of Physics, Graduate School of Science,Hokkaido University

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Dirac-Kähler Twist

Dirac-Kähler twist: a kind of topological twist

- applied for $\mathcal{N} = \#(\text{spinor components}) = 2^{D/2}$ SUSY in D-dimensional Euclidean space.

$$D = 2 \rightarrow \mathcal{N} = 2, \qquad D = 4 \rightarrow \mathcal{N} = 4, \ldots .$$

- defines tensor charges (twisted supercharges) of all rank,

$$Q_{i\alpha} \overset{\text{D-K twist}}{\Longrightarrow} Q, \; Q_{\mu}, \; Q_{\mu\nu}, \; \cdots .$$

The scalar charge Q can be interpreted as a BRST charge in a quantized TQFT.

- crucial to formulate lattice SUSY with

$$\#(\text{supercharges}) = \#(\text{doublers}) = 2^{D},$$
$$\text{i.e.} \qquad \mathcal{N} = 2^{D}/2^{D/2} = 2^{D/2}.$$

[D'Adda-Kanamori-Kawamoto-Nagata, Catterall et. al., Kaplan et. al., Sugino, . . .

Dirac-Kähler Twist & Lattice SUSY

Twisted lattice SUSY formalism

- $\mathcal{N} = D = 2 \rightarrow$ OK.
- $\mathcal{N} = D = 4 \rightarrow$ not yet. . . ,
  due to the lack of simple superspace formulation of $\mathcal{N} = D = 4$ SUSY in continuum.

Motivation

In order to construct lattice SUSY formalism, we need Dirac-Kähler twisted superspace formalism as its continuum counterpart, especially for $\mathcal{N} = D = 4$.

$\mathcal{N}=D=4$ Dirac-Kähler Twisted SUSY

$\mathcal{N}=D=4$ Dirac-Kähler TSYM (on-shell) has been analyzed so far. [Marcus, Kato-Kawamoto-Miyake, Blau-Thompson]

- The untwisted $\mathcal{N}=D=4$ SUSY has two $SO(4)$ symmetries:

$$[J_{\mu\nu}, Q_{i\alpha}] = \frac{1}{2}(\gamma_{\mu\nu})_{\alpha\beta} Q_{i\beta}, \qquad [R_{ab}, Q_{i\alpha}] = \frac{1}{2}(\gamma_{ab})_{ij} Q_{j\beta}.$$

- Define a new isometry:

$$\delta'_{\text{rot}} := \delta_{\text{Lorentz}} + \delta_{R\text{-symmetry}}, \qquad J'_{\mu\nu} := J_{\mu\nu} \otimes \mathbb{1} + \mathbb{1} \otimes R_{\mu\nu}.$$

- Take the Clebsch-Gordan decomposition

$$Q_{i\alpha} = \sum_p \frac{1}{p!} Q_{\mu_1\cdots\mu_p} (\gamma^{\mu_1\cdots\mu_p})_{i\alpha},$$

under which the supercharges transform as "Lorentz" tensors

$$\delta'_{\text{rot}} Q_{\mu_1\cdots\mu_p} = \sum_i \epsilon_{\mu_i[\mu} Q_{\mu_1\cdots\mu]\cdots\mu_p}.$$

Dirac-Kähler Twisted SUSY with Central Charge

This work

Yet another possible $\mathcal{N} = D = 4$ Dirac-Kähler twisted SYM:
the central charge extension. [Sohnius-Stelle-West]

- The untwisted $\mathcal{N} = D = 4$ SUSY w/ one central charge has the isometries

$$SO(4) \times USp(4) \;\; \underset{\uparrow}{\cong} \;\; SO(4) \times SO(5).$$

$$USp(4) \cong SO(5) \text{ (spinor repr.)}$$

- $SO(4)$ and $SO(5)$ are represented by the same set of gamma matrices
  $\longrightarrow$ we can carry out the D-K twisting as in the previous case.
- cf. [Yamron, Vafa-Witten]

Formulation

- We take the gauge-supercovariant derivative formulation:

$$D_A : \text{supercovariant derivative}$$
$$\rightarrow \nabla_A := D_A - i\Gamma_A : \text{gauge-supercovariant derivative.}$$

- Superfields are defined as the supercurvatures

$$\{\nabla_A, \nabla_B\} = -i\mathcal{F}_{AB},$$

to be constrained to make them irreducible.

- The constraints are found as follows:

$$\mathcal{N} = 1 \text{ 10D SYM} \xrightarrow{\text{dimensional reduction}} \mathcal{N} = 4 \text{ 5D SYM}$$
$$\xrightarrow{\text{Legendre transformation}} \mathcal{N} = 4 \text{ 4D SYM w/ CC}$$
$$\xrightarrow{\text{D-K twist}} \mathcal{N} = 4 \text{ 4D TSYM w/ CC.}$$

Dimensional Reduction through Legendre Transformation

- From 5D to 4D, we carry out a Legendre transformation to obtain a gauged central charge. [Sohnius-Stelle-West]

$$(\Phi,\ \partial_5\Phi) \to (\Phi,\ \Pi_\Phi), \qquad \mathcal{L}_{(5)} \to \mathcal{L},$$

$$\mathcal{L} := \mathcal{L}_{(5)} - \Pi_\Phi \partial_5 \Phi, \qquad \Pi_\Phi := \frac{\partial \mathcal{L}}{\partial(\partial_5 \Phi)} \quad \text{(conjugate to } \Phi\text{)},$$

so that

$$\begin{cases} \partial_5 \mathcal{L} = 0, & \text{no dependence on } x^5, \\ (\Phi,\ \Pi_\Phi), & \text{independent fields}, \\ P_5 \sim Z, & \text{central charge in 4D}. \end{cases}$$

These are on-shell in 5D, but off-shell in 4D:

$$\#(\text{fermions}) = 16,$$

$$\#(\text{bosons}) = 2\big(10\ (\text{10D gauge bosons}) - 1\ (\text{gauge}) - 1\ (\text{Lagrange multiplier})\big) = 16.$$

Constraints

We obtain the following constraints for the supercurvatures:

$\mathcal{N} = D = 4$ TSYM w/ CC constraints

$$
\begin{aligned}
&\{\nabla, \nabla\} = \{\tilde{\nabla}, \tilde{\nabla}\} = 0, && \{\nabla, \tilde{\nabla}\} = 2i(\nabla_5 - W),\\
&\{\nabla_{\mu\nu}, \nabla_{\rho\sigma}\} = 2i\varepsilon_{\mu\nu\rho\sigma}(\nabla_5 - W), && \{\tilde{\nabla}_{\mu\nu}, \tilde{\nabla}_{\rho\sigma}\} = 2i\varepsilon_{\mu\nu\rho\sigma}(\nabla_5 - W),\\
&\{\nabla_\mu, \nabla_\nu\} = \{\tilde{\nabla}_\mu, \tilde{\nabla}_\nu\} = 0, && \{\nabla_\mu, \tilde{\nabla}_\nu\} = 2i\delta_{\mu\nu}(\nabla_5 + W),\\
&\{\nabla, \nabla_{\mu\nu}\} = \{\tilde{\nabla}, \tilde{\nabla}_{\mu\nu}\} = 0, && \{\nabla, \tilde{\nabla}_{\mu\nu}\} = \{\tilde{\nabla}, \nabla_{\mu\nu}\} = 0,\\
&\{\nabla, \nabla_\mu\} = 2i(\nabla_\mu - W_\mu), && \{\tilde{\nabla}, \tilde{\nabla}_\mu\} = 2i(\nabla_\mu + W_\mu),\\
&\{\nabla, \tilde{\nabla}_\mu\} = \{\tilde{\nabla}, \nabla_\mu\} = 0, && \{\nabla_{\mu\nu}, \tilde{\nabla}_{\rho\sigma}\} = 2i\delta_{\mu\nu,\rho\sigma}(\nabla_5 - W),\\
&\{\nabla_{\mu\nu}, \nabla_\rho\} = 2i\delta_{\mu\nu,\rho}{}^\sigma(\nabla_\sigma + W_\sigma), && \{\tilde{\nabla}_{\mu\nu}, \tilde{\nabla}_\rho\} = 2i\delta_{\mu\nu,\rho}{}^\sigma(\nabla_\sigma - W_\sigma),\\
&\{\nabla_{\mu\nu}, \tilde{\nabla}_\rho\} = 2i\varepsilon_{\mu\nu,\rho}{}^\sigma(\nabla_\sigma - W_\sigma), && \{\tilde{\nabla}_{\mu\nu}, \nabla_\rho\} = 2i\varepsilon_{\mu\nu,\rho}{}^\sigma(\nabla_\sigma + W_\sigma).
\end{aligned}
$$

Independent Fields

Solving the constraints, we find the following independent fields:

Solutions—Independent Component Fields

$$\mathcal{F}_{\mu\nu} := [\nabla_\mu, W_\nu],\ \mathcal{F}_\mu := [\nabla_\mu, W],$$
$$H_\mu := [\nabla_5, W_\mu],\ H := [\nabla_5, W],\ g_\mu := [\nabla_\mu, \nabla_5],$$

$$\phi := W|, \qquad \phi_\mu := W_\mu|, \qquad \text{(5 scalars)}$$

$$\left.\begin{array}{c} \rho := \frac{1}{4}\mathcal{F}_{\mu,}{}^{\mu}|, \quad \tilde{\rho} := \frac{1}{4}\tilde{\mathcal{F}}_{\mu,}{}^{\mu}|, \quad \chi_{\mu\nu} := \frac{1}{2}\mathcal{F}_{[\mu,\nu]}|, \\ \lambda_\mu := \mathcal{F}_\mu|, \qquad \tilde{\lambda}_\mu := \tilde{\mathcal{F}}_\mu|, \end{array}\right\} \quad \text{(16 fermions)}$$

$$A_\mu := i\nabla_\mu|, \qquad \text{(1 gauge vector)}$$

$$H := \frac{i}{2}H|, \qquad H_\mu := \frac{i}{2}H_\mu|, \qquad \text{(5 auxiliary fields)}$$

$$V_\mu := \frac{i}{2}g_\mu|. \qquad \text{(1 pseudo vector, constrained)}$$

Supertransformations

Supertransformations are also given by solving the constraints. In general component filed $f = F|$, we obtain $Q_A f := [\nabla_A, F]|$.

Solutions—Supertransformations

$$
\begin{gathered}
Q\phi = -i\rho, \qquad \tilde{Q}\phi = -i\rho, \\
Q\phi_\mu = -i\lambda_\mu, \qquad \tilde{Q}\phi_\mu = +i\tilde{\lambda}_\mu, \\
Q\rho = 2H - [\nabla_\mu, \phi^\mu], \qquad \tilde{Q}\rho = 0, \\
Q\tilde{\rho} = 0, \qquad \tilde{Q}\tilde{\rho} = 2H + [\nabla_\mu, \phi^\mu], \\
Q\chi_{\mu\nu} = iF_{\mu\nu} - [\phi_\mu, \phi_\nu] + [\nabla_{[\mu}, \phi_{\nu]}], \\
\tilde{Q}\chi_{\mu\nu} = i\tilde{F}_{\mu\nu} - \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}[\phi_\rho, \phi_\sigma] - \varepsilon_{\mu\nu}{}^{\rho\sigma}[\nabla_\rho, \phi_\sigma], \\
Q\lambda_\mu = 0, \qquad \tilde{Q}\lambda_\mu = -2(V_\mu - H_\mu), \\
Q\tilde{\lambda}_\mu = -2(V_\mu + H_\mu), \qquad \tilde{Q}\tilde{\lambda}_\mu = 0,
\end{gathered}
$$

Supertransformations

Solutions—Supertransformations (cont'd)

$$QA_\mu = \lambda_\mu, \qquad \tilde{Q}A_\mu = \tilde{\lambda}_\mu,$$

$$QH = -\frac{i}{2}\left([\nabla_\mu, \lambda^\mu] + [\lambda_\mu, \phi^\mu]\right),$$

$$\tilde{Q}H = -\frac{i}{2}\left([\nabla_\mu, \tilde{\lambda}^\mu] - [\tilde{\lambda}_\mu, \phi^\mu]\right),$$

$$QH_\mu = \frac{i}{2}\left([\nabla_\mu, \tilde{\rho}] - [\nabla_\rho, \tilde{\chi}^\rho{}_\mu] - [\tilde{\rho}, \phi_\mu] + [\tilde{\chi}_\mu{}^\rho, \phi_\rho]\right) + i[\lambda_\mu, \phi],$$

$$\tilde{Q}H_\mu = -\frac{i}{2}\left([\nabla_\mu, \rho] - [\nabla_\rho, \chi^\rho{}_\mu] + [\rho, \phi_\mu] - [\chi_\mu{}^\rho, \phi_\rho]\right) + i[\tilde{\lambda}_\mu, \phi],$$

$$QV_\mu = -QH_\mu, \qquad \tilde{Q}V_\mu = +\tilde{Q}H_\mu,$$

and for the other charges (Q_μ, $\tilde{Q}_\mu$, $Q_{\mu\nu}$).

Action

Solutions—Off-shell Action

$$
\begin{aligned}
\mathcal{L} = \mathrm{Tr}\Big[& -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}V_\mu V^\mu + \frac{1}{2}H_\mu H^\mu + \frac{1}{2}HH \\
& - \frac{1}{4}[\nabla_\mu, \phi][\nabla^\mu, \phi] - \frac{1}{4}[\nabla_\mu, \phi_\nu][\nabla^\mu, \phi^\nu] \\
& + \frac{i}{2}\rho[\nabla_\mu, \lambda^\mu] + \frac{i}{2}\tilde{\rho}[\nabla_\mu, \tilde{\lambda}^\mu] + \frac{i}{2}\chi\mu\nu[\nabla_\nu, \lambda_\mu] + \frac{i}{2}\tilde{\chi}\mu\nu[\nabla_\nu, \tilde{\lambda}_\mu] \\
& - i\rho[\phi_\mu, \tilde{\lambda}^\mu] - i\tilde{\rho}[\phi_\mu, \lambda^\mu] + i\chi^{\mu\nu}[\phi_\nu, \tilde{\lambda}_\mu] + i\tilde{\chi}^{\mu\nu}[\phi_\nu, \lambda_\mu] \\
& - \frac{i}{2}\lambda_\mu[\phi, \lambda^\mu] + \frac{i}{2}\tilde{\lambda}_\mu[\phi, \tilde{\lambda}^\mu] - \frac{i}{2}\rho[\phi, \rho] + \frac{i}{2}\tilde{\rho}[\phi, \tilde{\rho}] \\
& - \frac{i}{8}\chi^{\mu\nu}[\phi, \chi_{\mu\nu}] + \frac{i}{8}\tilde{\chi}^{\mu\nu}[\phi, \tilde{\chi}\mu\nu] \\
& + \frac{1}{4}[\phi, \phi_\mu][\phi, \phi^\mu] + \frac{1}{4}[\phi_\mu, \phi_\nu][\phi^\mu, \phi^\nu]\Big]
\end{aligned}
$$

Manifest Invariance

- The action is manifestly invariant under Q and $\tilde{Q}$ as it is Q and $\tilde{Q}$-exact:
$$\mathcal{L} = QL = \tilde{Q}\tilde{L}.$$
- The other invariances are also shown manifestly, since
$$\begin{aligned} Q_{(\mu}L_{\nu)} &= \frac{1}{2}\delta_{\mu\nu}\mathcal{L} = \tilde{Q}_{(\mu}\tilde{L}_{\nu)}, \\ Q_{\mu\nu}L_{\rho\sigma} &= \frac{1}{12}\delta_{\mu\nu,\rho\sigma}\mathcal{L} \\ &\quad + Q_{\mu\rho}L_{\nu\sigma} - Q_{\mu\sigma}L_{\nu\rho} - Q_{\nu\rho}L_{\mu\sigma} + Q_{\nu\sigma}L_{\mu\rho}, \end{aligned}$$
where
$$Q^{\mu}L_{\mu} = \frac{1}{2}Q^{\mu\nu}L_{\mu\nu} = \mathcal{L},$$
which can be rewritten as
$$\mathcal{L} \sim Q_{\mu}L_{\mu} \sim \tilde{Q}_{\mu}\tilde{L}_{\mu} \sim Q_{\mu\nu}L_{\mu\nu} \qquad \text{no summation.}$$

Manifest Invariance

Explicit Formulas

$$L = \frac{1}{4}\left(\rho\tilde{Q}\tilde{\rho} - \tilde{\lambda}_\rho\tilde{Q}\lambda^\rho + \frac{1}{2}\tilde{\chi}_{\rho\sigma}\tilde{Q}\chi^{\rho\sigma}\right),$$

$$L_\mu = \frac{1}{16}\left(-\tilde{\rho}\tilde{Q}_\mu\rho + \tilde{\lambda}_\rho\tilde{Q}_\mu\lambda^\rho + \lambda_\rho\tilde{Q}_\mu\tilde{\lambda}^\rho - \frac{1}{2}\tilde{\chi}_{\rho\sigma}\tilde{Q}_\mu\chi^{\rho\sigma}\right),$$

$$L_{\mu\nu} = \frac{1}{24}\left(\tilde{\rho}\tilde{Q}_{\mu\nu}\rho + \rho\tilde{Q}_{\mu\nu}\tilde{\rho} - \tilde{\lambda}_\rho\tilde{Q}_{\mu\nu}\lambda^\rho - \lambda_\rho\tilde{Q}_{\mu\nu}\tilde{\lambda}^\rho + \frac{1}{2}\tilde{\chi}_{\rho\sigma}\tilde{Q}_{\mu\nu}\chi^{\rho\sigma}\right),$$

$$\tilde{L} = \frac{1}{4}\left(\tilde{\rho}Q\rho - \lambda_\rho Q\tilde{\lambda}^\rho + \frac{1}{2}\tilde{\chi}_{\rho\sigma}Q\chi^{\rho\sigma}\right),$$

$$\tilde{L}_\mu = \frac{1}{16}\left(-\rho Q_\mu\tilde{\rho} + \tilde{\lambda}_\rho Q_\mu\lambda^\rho + \lambda_\rho Q_\mu\tilde{\lambda}^\rho - \frac{1}{2}\tilde{\chi}_{\rho\sigma}Q_\mu\chi^{\rho\sigma}\right),$$

$$\tilde{L}_{\mu\nu} = \frac{1}{24}\left(\tilde{\rho}Q_{\mu\nu}\rho + \rho Q_{\mu\nu}\tilde{\rho} - \tilde{\lambda}_\rho Q_{\mu\nu}\lambda^\rho - \lambda_\rho Q_{\mu\nu}\tilde{\lambda}^\rho + \frac{1}{2}\tilde{\chi}_{\rho\sigma}Q_{\mu\nu}\chi^{\rho\sigma}\right).$$

Analysis

- The SUSY algebra closes off-shell by construction.
- But the off-shell multiplet contains a constraint field V_μ, with the constraint

$$[\nabla_\mu, V_\mu] + [W, H] + [W_\mu, H_\mu] + \{\rho, \tilde{\rho}\} + \{\lambda_\mu, \tilde{\lambda}_\mu\} + \frac{1}{2}\{\chi_{\mu\nu}, \tilde{\chi}_{\mu\nu}\} = 0.$$

  The solution of the constraint: nonlocal. [Siegel]
  Lagrange multiplier technique to impose the constraint
  $\rightarrow$ on-shell $\mathcal{N} = D = 4$ standard SYM multiplet.
- Topological aspects: all the twisted supercharges Q_A are nilpotent$\rightarrow$ BRST-like charges, with ghost number essentially comes from the chirality. Especially,

$$T_{\mu\nu} = \{Q, \Lambda_{\mu\nu}\} = \{\tilde{Q}, \tilde{\Lambda}_{\mu\nu}\} : \text{energy-momentum tensor.}$$

Summary

- In this talk, the exact off-shell formulation of $N = D = 4$ Dirac-Kähler-twisted SYM theory with a central charge is manifestly presented on the twisted superspace.
    - The invariance of the presented action under all of the sixteen twisted supercharges can be shown explicitly, as well as the fact that the twisted SUSY algebra closes off-shell.
    - The theory should be interpreted as a topological field theory as in the other twisted supersymmetric theories.
- The resultant theory could play an interesting role from, especially, the viewpoint of a lattice SUSY formulation.