The Growth of Negativity

Geometry, Topology, and Dimensionality in String theory

- ES hep-th/0510044, PRD 73 086004
- J. McGreevy ES, D. Starr hep-th/061xxxx
- D. Green, A. Lawrence, JMcG, D. Morrison, ES in progress

cf O. Aharony, ES hep-th/?

Consider moduli potential in a string compactification X of volume V

$$\underset{\substack{4d \\ \text{Einstein frame}}}{\mathcal{U}}(g_s, V, \ldots) = \frac{g_s^2 (D - D_c)}{V} + \frac{g_s^2}{V} \int_X \frac{\sqrt{G_X}(-R)}{V}$$

+ fluxes + orientifolds + branes + loops + non-perturbative

From the worldsheet point of view

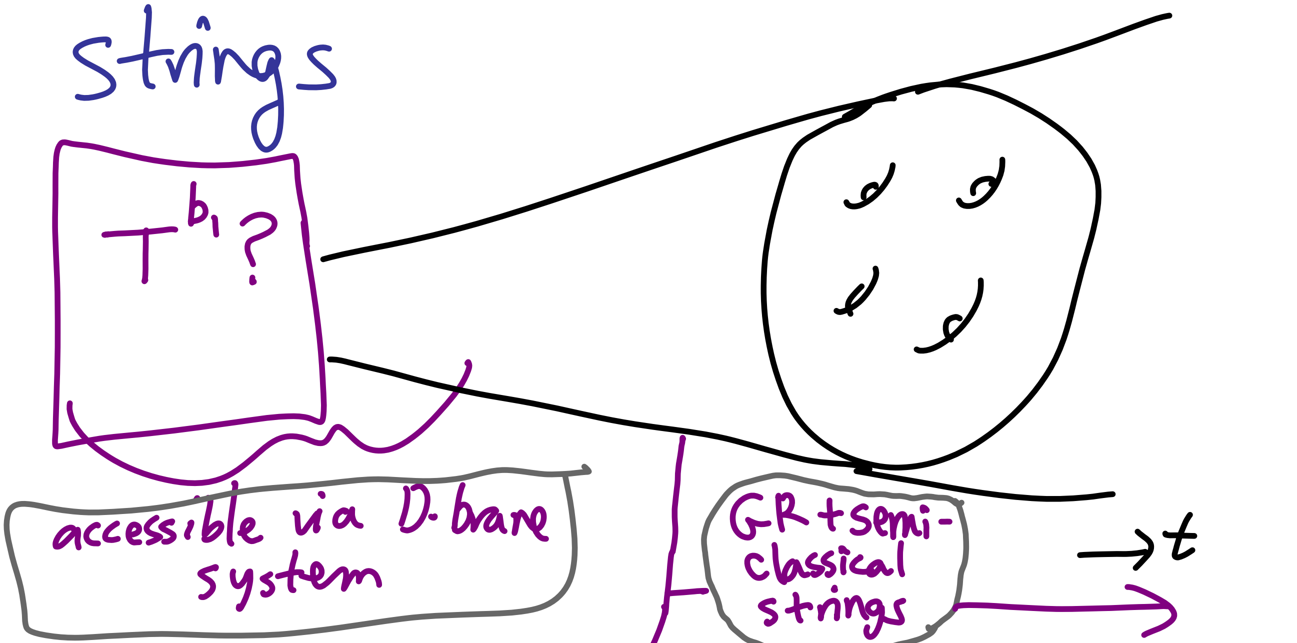
$$\beta_{\log g_{\text{eff}}^2} \sim D - D_c + \frac{\int -R}{V} + \cdots$$

↳ Raises the question:

Q Are negatively curved compactifications of 10d string theory effectively supercritical?

This talk:

Evidence for a new duality ("D duality") between "critical" strings on compact negatively curved space, and supercritical strings



Small radius: D-probes, AdS/CFT suggest natural transition to Jacobian torus

Large radius: exponential density of winding strings ⇒ "effective central charge" $> c_{crit}$

String theory on small spaces does interesting things:

- For example • T-duality: $S^1_R \cong S^1_{\frac{1}{R}}$
  - winding modes build up momentum modes on dual circle
  - But $\underline{c_{eff}}$ same for all R

$$\int \frac{d^2\tau}{\tau_2^2} Z_1(\tau) = \mathrm{Tr} \int \frac{d^2\tau}{4\tau_2} q^{L_0} \bar{q}^{\bar{L}_0}$$

$$Z_1 \xrightarrow[\text{UV}]{\tau_2 \to 0} \sim e^{c_{eff}\pi/6\tau_2}$$

c_{eff} counts net effective # of dimensions in which string can oscillate.

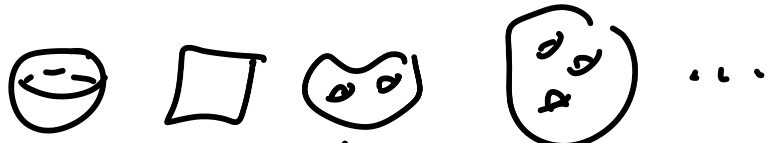
e.g. type 0 $c_{eff} = 12$, bosonic $c_{eff} = 24$, SUSY $c_{eff} = 0$

- Another example: small Calabi-Yau
  $\rightarrow$ LG phase, topology change, ...

Above was for flat target spaces.

<u>Curved</u> target spaces are generic.

e.g. $d=2$



S^2 T^2 $h=2$ $h=3$ ...

and at large radius, expand slowly with time <u>controlled</u> by GR e.g. constant curvature vacuum sol'n $-dt^2 + t^2 ds^2_{\mathbb{H}n/\Gamma}$

Relevant for basic physical questions:

- compactifications (most are curved)
- 4d space may be globally compact

↳ initial singularity, measure ?

- Intrinsic interest: correspondence between topology & effective dimensionality

Back to our question.

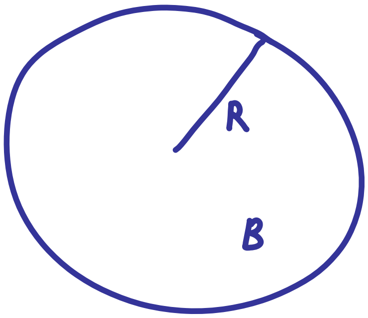
Are negatively curved compactifications of 10d string theory effectively supercritical?

Answer: Yes, for compact spaces M_n with negative sectional curvature.

because there is an exponential density of winding modes!

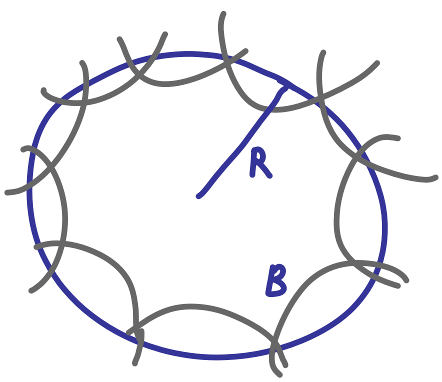
- Milnor '68: $\pi_1(M_n)$ has exponential growth (in "word metric")
- Margulis '69: the number of closed geodesics (up to homotopy) grows exponentially. $\rho(L) \propto e^{L/L_0}$

The covering space $\tilde{M}_n$ has



$$\mathrm{Vol}\big(B(R)\big) \propto e^{(n-1)\frac{R}{R_0}}$$

$M_n = \tilde{M}_n / \Gamma$ compact $\Rightarrow$



exponential # of closed geodesics of length L

Why is exponential growth relevant?
It leads to a Hagedorn density of winding strings and a new contribution to C_{eff}:

$$\rho(m) = e^{m\sqrt{\alpha'}\,\pi\sqrt{\frac{2C_{eff}}{3}}}$$

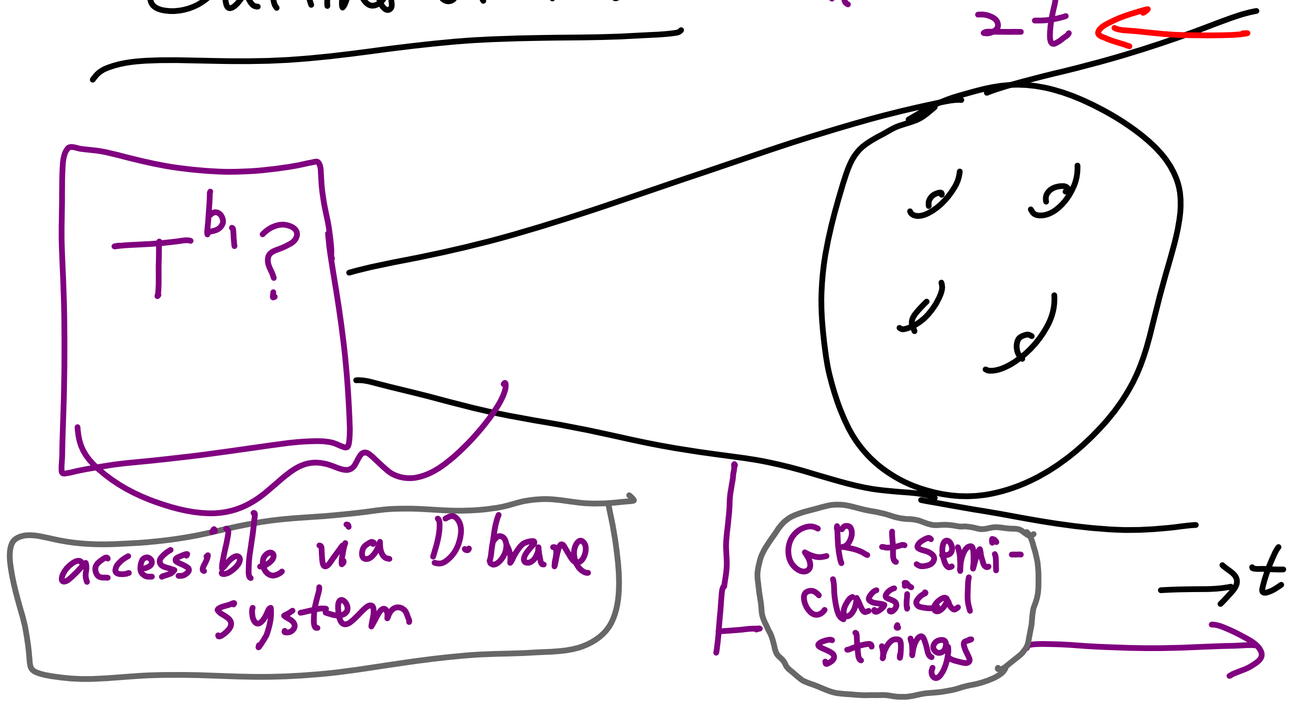
$$\underbrace{\int dm\, \rho(m)\, e^{-\pi \tau_2 \alpha' m^2}}_{\text{saddle point}} \underset{\tau_2 \to 0}{\sim} e^{\frac{\pi C_{eff}}{6\tau_2}}$$

$$2 m_* \pi \tau_2 \alpha' = \sqrt{\alpha'}\,\pi\sqrt{\frac{2C_{eff}}{3}}$$

So exponential growth of (conjugacy classes of) π_1 will yield a contribution to C_{eff} from the sum over winding modes.

Outline of rest

$C_{eff} = \frac{3\,g'(n-1)^2}{2t}$



① Computation of C_{eff} and check of Modular invariance for t-dependent $M_n = \mathbb{H}_n/\Gamma$

$$\underset{(UV)}{C_{eff}} \xleftrightarrow[\tau \to -\frac{1}{\tau}]{} \underset{(IR)}{\text{pseudo-tachyons}}$$

② Brief comments on generalizations

③ very small ra ius and the Jacobian/Albanese variety (T^{b_1}) via AdS/CFT

A consistent worldsheet CFT must satisfy 1-loop modular invariance.

Recall standard static cases

$$\int \frac{d^2\tau}{\tau_2^2} \underbrace{Z_1(\tau)}_{= Z_1(-\frac{1}{\tau})} = \mathrm{Tr} \int \frac{d^2\tau}{4\tau_2} q^{L_0} \bar{q}^{\bar{L}_0}$$

$$Z_{IR} \sim e^{-\pi \tau_2 \alpha' m_{min}^2} \qquad \tau_2 \to \infty$$

$$Z_{UV} \sim e^{-\frac{\pi \alpha' m_{min}^2}{\tau_2}} = e^{\frac{\pi c_{eff}}{6\tau_2}} \qquad \tau_2 \to 0$$

$\Rightarrow$ $c_{eff} \neq 0$ means IR divergence in path integral (Kutasov, Seiberg). However, in time dependent context, the instability does not always cause significant back reaction.

(Aharony, ES; cosmo theorists, early timelike linear dilaton works, ...)

Consider compact hyperbolic manifold $M_n = \mathbb{H}_n / \Gamma$ $\leftarrow ds^2 = dy^2 + \sinh^2 y \, d\Omega$

$$ds^2 = -dt^2 + t^2 ds^2_{\begin{cases}\mathbb{H}_n \\ \mathbb{H}_n/\Gamma \equiv M_n \text{ compact}\end{cases}} \Big\} \mathbb{R}^{n,1} \text{ (negative curvature FRW)}$$

$$+ ds^2_{\perp, 9-n} \Big\} \text{ some Ricci-flat space with moduli}$$

Satisfies Einstein's equations = leading order worldsheet $\beta = 0$ equations

↪ time-dependent worldsheet CFT

Modular invariance $\Rightarrow$

$$\begin{cases} \mathbb{H}_n \text{ theory} : c_{eff} = 0 \Leftrightarrow m^2_{min} = 0 & \text{gap in spatial modes} \\ \mathbb{H}_n/\Gamma \text{ theory} : c_{eff} \neq 0 \Leftrightarrow m^2_{min} < 0 & \text{zero mode normalizable} \end{cases}$$

A check of modular invariance for $M_n = \mathbb{H}_n / \Gamma$:

First consider covering space $(\mathbb{R}^{n,1})$

$$ds^2 = -dt^2 + t^2 ds^2_{\mathbb{H}_n} + ds^2_{\perp}$$

UV $\quad c_{eff} = 0$ (SUSY cancellation)

IR $\quad m^2_{min} = 0$: gapped spectrum on $\mathbb{H}_n$ spatial slices, compensated by t-dependence, as follows.

$$ds^2_{\mathbb{H}_n} = dy^2 + \sinh^2 y \, d\Omega$$

$$\nabla^2 \eta = -k^2 \eta$$

Normalizability: $\int^{\infty} \sqrt{G} \, f^*(y) f(y) = 1 \Rightarrow \boxed{k^2 > \frac{(n-1)^2}{4}}$ gap

The full t-dependent Laplacian is

$$\mathcal{H} = \nabla^2 = \frac{1}{t^2}\left(-\left(t^2\partial_t^2 + n t \partial_t\right) + \nabla^2_{\mathbb{H}_n}\right)$$

Work in basis of modes satisfying

$$\nabla^2 \eta = \frac{\omega^2 - k^2 + \frac{(n-1)^2}{4}}{t^2}\,\eta \qquad \eta = t^{\frac{1-n}{2}} e^{i\omega \ln t} f_{k,L}(y)\, Y_L(\Omega)$$

$\rightarrow$ IR limit of partition function is

$$\int d^d x \sqrt{-g}\, \Lambda(t) = \int d^d x \sqrt{-g}\, \mathrm{Tr} \log(\mathcal{H})$$

$$= \int dt\, dy\, d\Omega\, t^n \sinh^{n-1} y\, V_{S^{n-1}} \int \frac{d\tilde{\omega}}{2\pi} \sum_{L,k^2 > \frac{(n-1)^2}{4}} f^* f\, Y^* Y$$

$$\times \int \frac{d\tau_2}{\tau_2}\, e^{-\pi\alpha' \frac{\tau_2}{t^2}\underbrace{\left(\tilde{\omega}^2 + k^2 - \frac{(n-1)^2}{4}\right)}_{}}$$

$m^2_{min} = 0$ as it must be for flat space

So far recovered expected behavior for flat $\mathbb{R}^{n,1}$ (sliced by $\mathbb{H}_n$).

We can now see the effect of the projection Γ: $M_n = \mathbb{H}_n/\Gamma$

Modular invariance requires an IR divergence as $\tau_2 \to \infty$, since $c_{\text{eff}} \neq 0$

IR: no gap in spatial spectrum (k=0 mode is normalizable)

$$\Rightarrow \quad m^2_{\min}(t) = \frac{-(n-1)^2}{4t^2} < 0$$

$$Z_1\Big|_{\tau_2\to\infty} \sim \int dt\, t^n e^{\pi\alpha'\tau_2 \frac{(n-1)^2}{4t^2}}$$

✓

This is consistent with modular invariance, but does $m^2_{\min} < 0 \Rightarrow$ catastrophic instability?

cf Kutasov, Seiberg

No : This 0-mode with $m^2_{min} = -\frac{(n-1)^2}{4t^2}$ condenses, but does not cause significant back reaction:

$$S_0 = \int dt\, t^n\, \dot{\eta}^2$$

Solutions $\langle \eta \rangle = \eta_0 + \frac{\eta_1}{t^{n-1}}$

$\Rightarrow$ • energy density $\rho_\eta \sim \frac{1}{t^{2n}} \ll R \sim \frac{1}{t^2}$

so negligible effect on metric

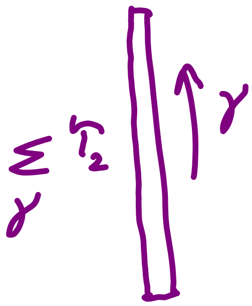
• Hubble friction damps η motion

• $\langle \eta \rangle$ doesn't change C_{eff}

Many smooth t-dependent backgrnds have this feature (e.g. inflationary density perturbations): "pseudotachyons"

(... O. Aharony & ES)

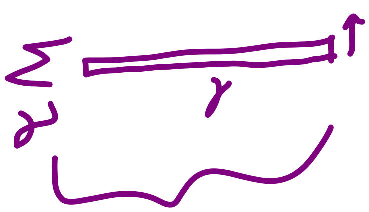
The IR & UV limits of Z_1 for the compact space $M_n = \mathbb{H}_n/\Gamma$ are thus



$$Z_1\Big|_{\tau_2\to\infty} \sim \int dt\, t^n e^{\pi\alpha'\tau_2\frac{(n-1)^2}{4t^2}}$$

pseudo-tachyon

Modular invariance $\Rightarrow$



$$Z_1\Big|_{\tau_2\to 0} \sim \int dt\, t^n e^{\frac{\pi\alpha'(n-1)^2}{4t^2\,\tau_2}}$$

winding modes (exponential growth)

$$\Rightarrow \boxed{c_{eff} = \frac{3\alpha'(n-1)^2}{2t^2}}$$

Can indeed write $s\to\infty$ limit of path integral as sum over windings (Selberg, Gutzwiller ...)

cf Berkooz, Pioline; Ooguri, Maldacena ...

$$\int d\ell \frac{1}{(\sinh\frac{\ell}{2})^{n-1}}\, \rho(\ell)\, e^{-\frac{\alpha'\ell^2}{4\pi}\tau_2}$$

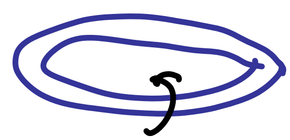
$\rho(\ell) \sim e^{(n-1)\frac{\ell}{t}}$ (cf Margulis)

SUSY structure

If we apply this in the "critical" superstring, must address issue of (asymptotic) SUSY.

- Fixed winding, $L_0 \to \infty$: SUSY breaking sensed by paths around nontrivial cycles introduced by orbifolding, suppressed by e^{-tm}

- $\ell \to \infty$, L_0 fixed: ~~SUSY~~ sensed directly



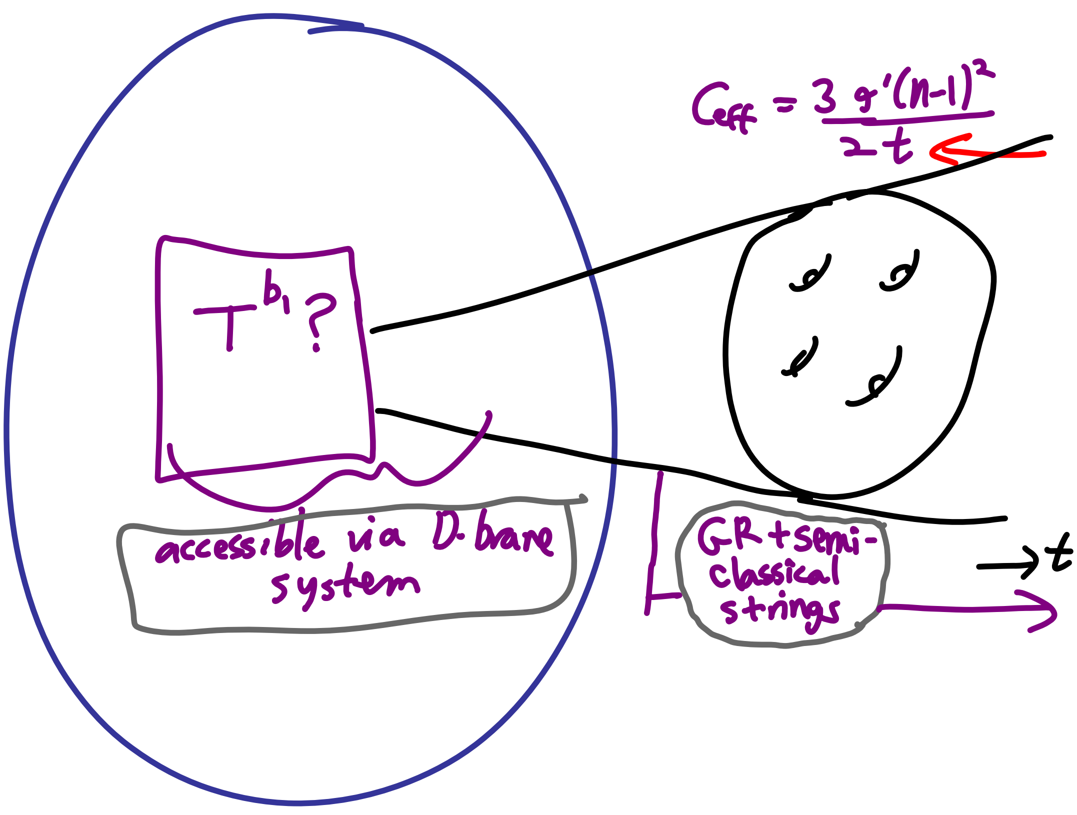
Also, enhanced IR & UV divergences in the bosonic case require the winding string contribution.

More generally, string backgrounds with sectional curvature < 0 are effectively supercritical

$$
\begin{tikzcd}
\text{Geometry} \arrow[r, leftrightarrow, "\text{Milnor, ...}"] \arrow[d, Rightarrow] & \text{Topology} \arrow[d, Rightarrow] \\
\begin{array}{l}\text{pseudotachyonic}\\ \text{perturbation}\\ \text{spectrum}\end{array} \arrow[r, leftrightarrow, "q \leftrightarrow -\frac{1}{q}"] & c_{eff} > 0 \text{ dimensionality}
\end{tikzcd}
$$

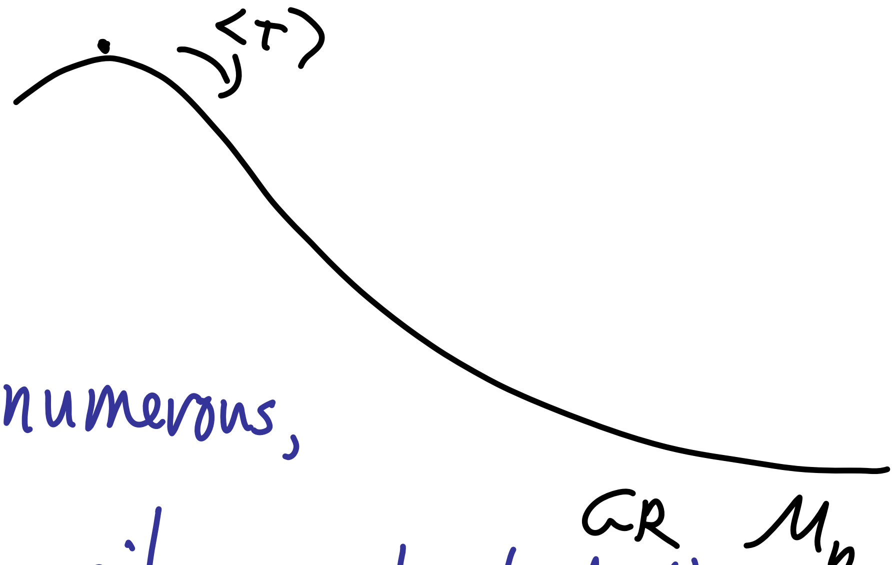
- as M_n shrinks, how large does c_{eff} get?
- Is there a way to control this regime?

No universal answer, but there is a natural starting point, Moreover one which appears to be amenable to AdS/CFT like definition



There are many ways to UV complete the nonlinear sigma model on $\mathcal{M}_n$

e.g. gauged linear σ-model ((2,2) SUSY) describing embedding of $\mathcal{M}_n$ in higher-D space, with a tachyon (relevant perturbation) effecting the transition.



Though numerous,

Models <u>easily</u> constructed this way are <u>not</u> generic ...

... e.g. in case M_n = Riemann surface,
The Jacobian torus may provide a much more natural starting point. For starters, it

- is classical moduli space of D-probe

hol. 1-forms ω_i $i = 1 \dots h$

$$\left(\int_{p_0}^{p} \omega_1, \dots, \int_{p_0}^{p} \omega_h \right) \text{ traces out } T^{2h}$$

- is singled out Mathematically:

$$\begin{array}{ccc} T^n & \longrightarrow & \text{Jacobian} \\ & & \downarrow \\ & & \Sigma_h \end{array}$$

- is singled out physically by symmetries: winding charge preserved by this map (1-cycles) → (1-cycles)

- Long wound strings trace out a lattice random walk in $2h$ dimensions.

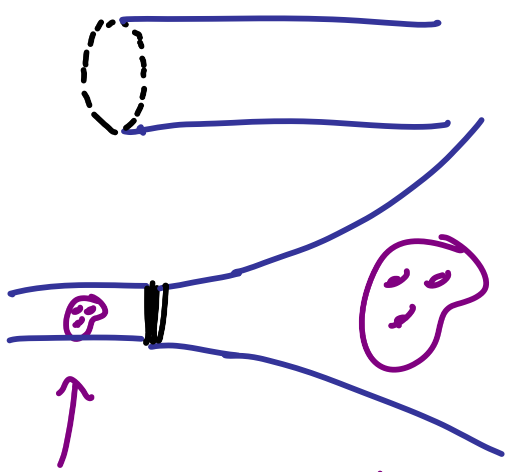
Toward "D duality":

We find an AdS/CFT setup where a small negatively curved space (= supercritical background of string theory) seems definable via a dual large-N QFT whose IR regime corresponds to the small M_n and has approximate moduli space $(T^{b_1})^N/S_N$

Take e.g. $AdS_5 / \mathcal{N}=4$ (S)U(N) SYM on Poincare patch, slice the $\mathbb{R}^{3,1}$ slices with $\mathbb{H}_3$, and mod by Γ to obtain compact space

$$ds_0^2 = \left(\frac{r}{\ell}\right)^2 \Big(-dt^2 + t^2 ds^2_{M_3 = \mathbb{H}_3/\Gamma}\Big) + \frac{\ell^2 dr^2}{r^2} + d\Omega_5$$

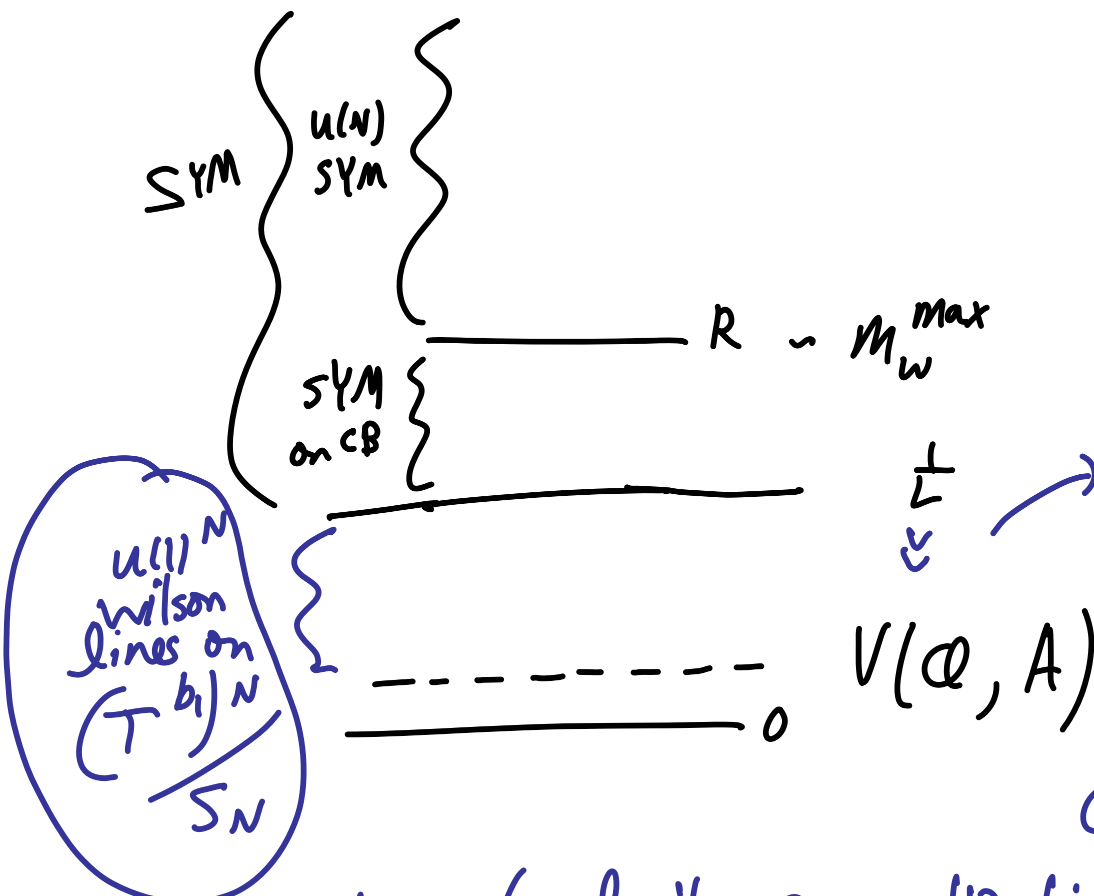
Go out on the approximate Coulomb branch via shell of D3-branes at $r = R$ cf Kraus et al, Horowitz ES



$$\left.\begin{array}{l} S^5 \\ AdS_5 \end{array}\right\} \cong \left\{\begin{array}{l} \mathcal{N}=4\text{ SYM} \\ \text{on } -dt^2 + t^2 ds^2_{\mathbb{H}_3/\Gamma} \\ \text{on approximate}^* \\ \text{Coulomb branch} \end{array}\right.$$

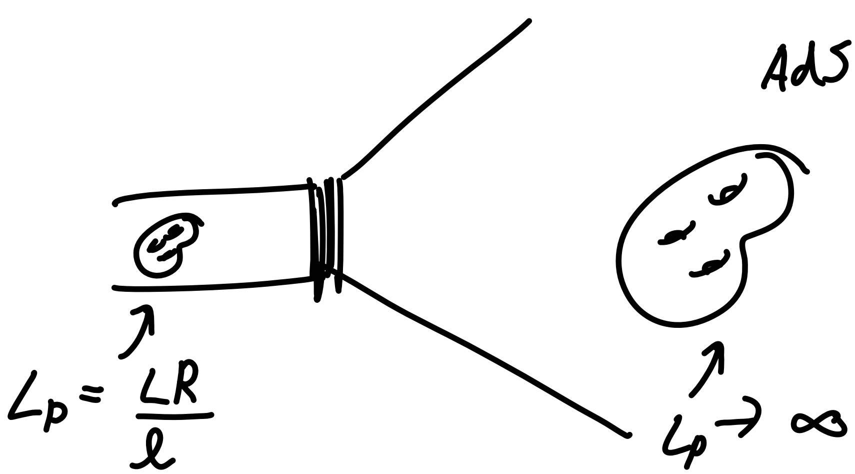
- Hubble friction → slow motion
- off-diagonal modes massive → suppressed corrections

In the QFT we have the energy scales



* potential for wilson lines and scalars is suppressed on approximate Coulomb branch

Jacobian (and its generalizations) appear naturally in IR of QFT...



Conclusion

- Negative curvature & compactness $\Rightarrow$ Rich topology $\Rightarrow$ higher effective dimensionality (exponential density of winding strings ; T^{b_1} in D-brane systems ...)

Novel mechanism for growing dimensions; suggests "D duality" generalizing T duality

- AdS/CFT provides formulation, including small-volume regime

↳ close to standard supercritical theory on T^{b_1} ? Remains to be analyzed ...