

Supersymmetric two-dimensional QCD at finite temperature

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Outline

- light-cone coordinates
- (supersymmetric) discrete light-cone quantization
- $\mathcal{N} = 1$ SQCD-CS₁₊₁
[NBP 661, 99 (2003); PRD 67, 115005 (2003)]
- thermodynamics
[PRD 70, 065012 (2004)]
- Lanczos algorithm for density of states
- preliminary results
- future work

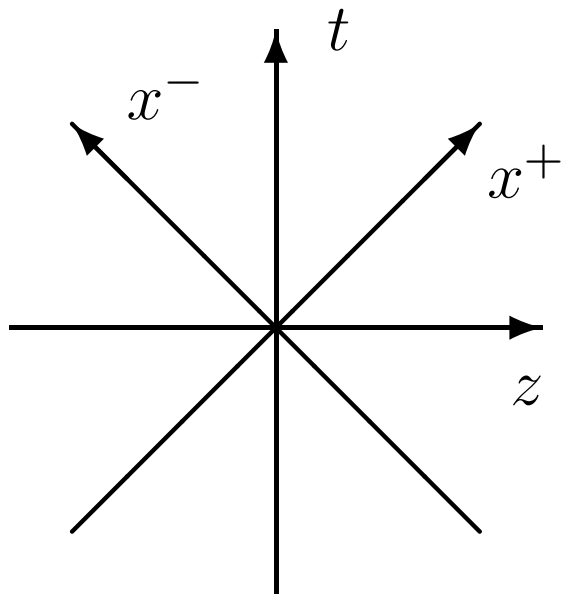
Acknowledgments

- work done in collaboration with
 - S. Pinsky, Y. Proestos, N. Salwen, and U. Trittmann.
- supported in part by
 - US Department of Energy.
 - Minnesota Supercomputing Institute.

Light-cone coordinates

Dirac, RMP **21**, 392 (1949).

- time: $x^+ = t + z$
- space: $\underline{x} = (x^-, \vec{x}_\perp)$, $x^- \equiv t - z$, $\vec{x}_\perp = (x, y)$
- energy: $p^- = E - p_z$
- momentum: $\underline{p} = (p^+, \vec{p}_\perp)$, $p^+ \equiv E + p_z$, $\vec{p}_\perp = (p_x, p_y)$
- mass-shell condition: $p^2 = m^2 \Rightarrow p^- = \frac{m^2 + p_\perp^2}{p^+}$



Discrete light-cone quantization (DLCQ)

Pauli and Brodsky, PRD **32**, 1993 (1985); 2001 (1985).
Brodsky, Pauli, and Pinsky, Phys. Rep. **301**, 299 (1997).

- light-cone box $-L < x^- < L$, $-L_\perp < x, y < L_\perp$
and periodic BC
→ discrete grid: $p_i^+ \rightarrow \frac{\pi}{L} n_i$, $\mathbf{p}_{i\perp} \rightarrow \left(\frac{\pi}{L_\perp} n_{ix}, \frac{\pi}{L_\perp} n_{iy} \right)$.
- limit $L \rightarrow \infty$ exchanged for limit in terms of the integer
resolution $K \equiv \frac{L}{\pi} P^+$ for fixed total momentum P^+ .
- $n_i > 0 \Rightarrow \#$ of particles $\leq K$.
- integrals replaced by discrete sums

$$\int dp^+ \int d^2 p_\perp f(p^+, \mathbf{p}_\perp)$$

$$\simeq \frac{2\pi}{L} \left(\frac{\pi}{L_\perp} \right)^2 \sum_n \sum_{n_x, n_y = -N_\perp}^{N_\perp} f(n\pi/L, \mathbf{n}_\perp \pi/L_\perp).$$

Supersymmetric DLCQ (SDLCQ)

Matsumura, Sakai, and Sakai, PRD **52**, 2446 (1995).
Lunin and Pinsky, AIP Conf. Proc. **494**, 140 (1999).

- supersymmetry algebra: $\{Q^+, Q^+\} = 2\sqrt{2}P^+$,
 $\{Q^-, Q^-\} = 2\sqrt{2}P^-$, $\{Q^+, Q^-\} = -4P_\perp$.
- discretize supercharge Q^- and compute
 $P_{\text{SDLCQ}}^- = \frac{1}{2\sqrt{2}} \{Q^-, Q^-\} \neq P_{\text{DLCQ}}^-$
- preserves supersymmetry.
- for ordinary DLCQ, recover supersymmetry only in infinite resolution limit.
- primarily work in large- N_c approximation.

Mesons and glueballs

- Either could be a boson or a fermion.
- Meson Fock state:

$$\bar{f}_{i_1}^\dagger(k_1) a_{i_1 i_2}^\dagger(k_2) \dots b_{i_n i_{n+1}}^\dagger(k_{n-1}) \dots f_{i_p}^\dagger(k_n) |0\rangle$$

- Glueball Fock state: $\text{Tr}[a_{i_1 i_2}^\dagger(k_1) \dots b_{i_n i_{n+1}}^\dagger(k_n)] |0\rangle$

\bar{f}_i^\dagger and $f_i^\dagger \longrightarrow$ fundamental partons

a_{ij}^\dagger and $b_{ij}^\dagger \longrightarrow$ adjoint partons

- Work in large- N_c approximation.
 - no mixing, single-trace states only, baryons difficult
- SDLCQ can handle finite N_c with increased computing resources.

$\mathcal{N} = 1$ SQCD-CS₂₊₁

Action:

$$S = \int d^3x \text{Tr} \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \xi^\dagger D^\mu \xi + i \bar{\Psi} D_\mu \Gamma^\mu \Psi \right. \\ \left. -g \left[\bar{\Psi} \Lambda \xi + \xi^\dagger \bar{\Lambda} \Psi \right] + \frac{i}{2} \bar{\Lambda} \Gamma^\mu D_\mu \Lambda \right. \\ \left. + \frac{\kappa}{2} \epsilon^{\mu\nu\lambda} \left[A_\mu \partial_\nu A_\lambda + \frac{2i}{3} g A_\mu A_\nu A_\lambda \right] + \kappa \bar{\Lambda} \Lambda \right\}$$

Adjoint fields: gauge boson A_μ (gluons)

Majorana fermion Λ (gluinos).

Fundamental fields: Dirac fermion Ψ (quarks)

complex scalar ξ (squarks).

Chern–Simons coupling: $\kappa \longrightarrow$ mass for adjoint fields.

Transformations

- Covariant derivatives:

$$\begin{aligned} D_\mu \Lambda &= \partial_\mu \Lambda + ig[A_\mu, \Lambda], & D_\mu \xi &= \partial_\mu \xi + igA_\mu \xi, \\ D_\mu \Psi &= \partial_\mu \Psi + igA_\mu \Psi. \end{aligned}$$

- Supersymmetry transformations:

$$\begin{aligned} \delta A_\mu &= \frac{i}{2} \bar{\varepsilon} \Gamma_\mu \Lambda, & \delta \Lambda &= \frac{1}{4} F_{\mu\nu} \Gamma^{\mu\nu} \varepsilon, \\ \delta \xi &= \frac{i}{2} \bar{\varepsilon} \Psi, & \delta \Psi &= -\frac{1}{2} \Gamma^\mu \varepsilon D_\mu \xi. \end{aligned}$$

- Reduce to 1+1 dimensions by assuming fields independent of the transverse coordinate x^2 .

Supercharge

$$\begin{aligned}\bar{\varepsilon}Q = & \int dx^- dx^2 \left(\frac{i}{4} \bar{\varepsilon} \Gamma^{\alpha\beta} \Gamma^+ \text{tr} (\Lambda F_{\alpha\beta}) \right. \\ & + \frac{i}{2} D_- \xi^\dagger \bar{\varepsilon} \Psi + \frac{i}{2} \xi^\dagger \bar{\varepsilon} \Gamma^{+\nu} D_\nu \Psi \\ & \left. - \frac{i}{2} \bar{\Psi} \varepsilon D^+ \xi + \frac{i}{2} D_\nu \bar{\Psi} \Gamma^{+\nu} \varepsilon \xi \right) .\end{aligned}$$

Dynamical fields

Fermion components:

$$\Lambda = \left(\lambda, \tilde{\lambda} \right)^T, \quad \Psi = \left(\psi, \tilde{\psi} \right)^T, \quad Q = \left(Q^+, Q^- \right)^T.$$

Constraints in light-cone gauge ($A^+ = 0$):

$$\partial_- \tilde{\lambda} = -\frac{ig}{\sqrt{2}} \left([A^2, \lambda] + i\xi\psi^\dagger - i\psi\xi^\dagger \right),$$

$$\partial_- \tilde{\psi} = -\frac{ig}{\sqrt{2}} A^2 \psi + \frac{g}{\sqrt{2}} \lambda \xi - \kappa \lambda / \sqrt{2}, \quad \partial_-^2 A^- = gJ,$$

with

$$J \equiv i[A^2, \partial_- A^2] + \frac{1}{\sqrt{2}} \{ \lambda, \lambda \} + \kappa \partial_- A^2 - i h \partial_- \xi \xi^\dagger + i \xi \partial_- \xi^\dagger + \sqrt{2} \psi \psi^\dagger.$$

Reduced supercharge

$$Q^- = g \int dx^- \left\{ 2^{3/4} \left(i[A^2, \partial_- A^2] - \kappa \partial_- A^2 + \frac{1}{\sqrt{2}} \{\lambda, \lambda\} \right) \frac{1}{\partial_-} \lambda \right. \\ \left. - \frac{1}{\sqrt{2}} \left(i\sqrt{2} \xi \partial_- \xi^\dagger - i\sqrt{2} \partial_- \xi \xi^\dagger + 2\psi \psi^\dagger \right) \frac{1}{\partial_-} \lambda \right. \\ \left. - 2 \left(\xi^\dagger A^2 \psi + \psi^\dagger A^2 \xi \right) \right\} .$$

Mode expansions

$$A_{ij}^2(0, x^-) = \frac{1}{\sqrt{4\pi}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left(a_{ij}(k) e^{-ik\pi x^-/L} + a_{ji}^\dagger(k) e^{ik\pi x^-/L} \right),$$

$$\lambda_{ij}(0, x^-) = \frac{1}{2^{\frac{1}{4}} \sqrt{2L}} \sum_{k=1}^{\infty} \left(b_{ij}(k) e^{-ik\pi x^-/L} + b_{ji}^\dagger(k) e^{ik\pi x^-/L} \right),$$

$$\xi_i(0, x^-) = \frac{1}{\sqrt{4\pi}} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left(c_i(k) e^{-ik\pi x^-/L} + \tilde{c}_i^\dagger(k) e^{ik\pi x^-/L} \right),$$

$$\psi_i(0, x^-) = \frac{1}{2^{\frac{1}{4}} \sqrt{2L}} \sum_{k=1}^{\infty} \left(d_i(k) e^{-ik\pi x^-/L} + \tilde{d}_i^\dagger(k) e^{ik\pi x^-/L} \right).$$

Commutation relations

For finite N_c :

$$\left[a_{ij}, a_{kl}^\dagger \right] = \left(\delta_{il} \delta_{kj} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \right) ,$$

$$\left\{ b_{ij}, b_{kl}^\dagger \right\} = \left(\delta_{il} \delta_{kj} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \right) ,$$

$$\left[c_i, c_j^\dagger \right] = \delta_{ij} , \quad \left[\tilde{c}_i, \tilde{c}_j^\dagger \right] = \delta_{ij} ,$$

$$\left\{ d_i, d_j^\dagger \right\} = \delta_{ij} \quad \left\{ \tilde{d}_i, \tilde{d}_j^\dagger \right\} = \delta_{ij} .$$

Additional (Z_2) symmetry

Kutasov, NPB 414, 33 (1994).

$$a_{ij}(k, n^\perp) \rightarrow -a_{ji}(k, n^\perp) \quad b_{ij}(k, n^\perp) \rightarrow -b_{ji}(k, n^\perp)$$

Divides between even and odd numbers of gluons.

Diagonalize in each sector.

Finite temperature

- partition function: $Z = e^{-p_0/T}$
- one-dimensional bosonic free energy

$$\mathcal{F}_B = \frac{VT}{\pi} \sum_{n=1}^{\infty} \int_{M_n}^{\infty} dp_0 \frac{p_0}{\sqrt{p_0^2 - M_n^2}} \ln \left(1 - e^{-p_0/T} \right).$$

- fermionic free energy

$$\mathcal{F}_F = -\frac{VT}{\pi} \sum_{n=1}^{\infty} \int_{M_n}^{\infty} dp_0 \frac{p_0}{\sqrt{p_0^2 - M_n^2}} \ln \left(1 + e^{-p_0/T} \right).$$

Total free energy

- expand logs and integrate over p_0

$$\mathcal{F}(T, V) = -\frac{(K-1)\pi}{4}VT^2 - \frac{2VT}{\pi} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} M_n \frac{K_1\left((2l+1)\frac{M_n}{T}\right)}{(2l+1)}$$

- the sum over l is well approximated by the first few terms.
- represent the sum over n as an integral over a density of states: $\sum_n \rightarrow \int \rho(M)dM$
- approximate ρ by a continuous function.
- compute $\int dM$ by standard numerical techniques.

Lanczos algorithm for density of states

- Density of states: $\rho(M^2) = \sum_n d_n \delta(M^2 - M_n^2)$
where d_n is the degeneracy of the mass eigenvalue M_n .
- Cumulative distribution function:
 $N(M^2) = \int^{M^2} dM^2 \rho(M^2)$.
- Density written in form of a trace over $e^{-iP^- x^+}$:

$$\begin{aligned}\rho(M^2) &= \frac{1}{2P^+} \sum_n d_n \delta(M^2/2P^+ - P_n^-) \\ &= \frac{1}{4\pi P^+} \int_{-\infty}^{\infty} e^{iM^2 x^+/2P^+} \sum_n d_n e^{-iP_n^- x^+} dx^+ \\ &= \frac{1}{4\pi P^+} \int_{-\infty}^{\infty} e^{iM^2 x^+/2P^+} \text{Tr} e^{-iP^- x^+} dx^+.\end{aligned}$$

Estimation of the trace

- Approximate the trace as an average over a random sample of vectors
R. Alben, M. Blume, H. Krakauer, and L. Schwartz, PRB 12, 4090 (1975).
A. Hams and H. De Raedt, PRE 62, 4365 (2000).
- Define a local density for a single vector $|s\rangle$ as

$$\rho_s(M^2) = \frac{1}{4\pi P^+} \int_{-\infty}^{\infty} e^{iM^2 x^+ / 2P^+} \langle s | e^{-iP^- x^+} | s \rangle dx^+,$$

so that the average can be written

$$\rho(M^2) \simeq \frac{1}{S} \sum_{s=1}^S \rho_s(M^2).$$

Random phase vectors

T. Itaka and T. Ebisuzaki, PRE **69**, 057701 (2004).

- The sample eigenstates $|s\rangle$ chosen as random phase vectors.
- The coefficient of each Fock state in the basis is a random number of modulus one.

Approximation of matrix element

J. Jaklič and P. Prelovšek, PRB 49, 5065(R) (1994);
M. Aichhorn, M. Daghofer, H.G. Evertz, and W. von der Linden, PRB 67, 161103(R) (2003).

- Approximate $\langle s | e^{-iP^- x^+} | s \rangle$ by Lanczos iteration.
- Let D be the length of $|s\rangle$, and define $|u_1\rangle = \frac{1}{\sqrt{D}} |s\rangle$ as the initial Lanczos vector. Then

$$\rho_s(M^2) = \frac{D}{4\pi P^+} \int e^{iM^2 x^+ / 2P^+} \langle u_1 | e^{-iP^- x^+} | u_1 \rangle dx^+.$$

- $\langle u_1 | e^{-iP^- x^+} | u_1 \rangle$ can be approximated by the (1, 1) element of the exponentiation of the Lanczos tridiagonalization of P^- .

Exponentiation

- Let P_s^- be the tridiagonal Lanczos matrix and solve

$$P_s^- \vec{c}_n^s = \frac{M_{sn}^2}{2P^+} \vec{c}_n^s.$$

- A diagonal matrix $\Lambda_{ij} = \delta_{ij} \frac{M_{sn}^2}{2P^+}$ is related by the usual similarity transformation $P_s^- = U \Lambda U^{-1}$, where $U_{ij} = (c_j^s)_i$.
- The (1, 1) element is given by

$$\left(e^{-iP_s^- x^+} \right)_{11} = \sum_n |(c_n^s)_1|^2 e^{-iM_{sn}^2 x^+ / 2P^+}.$$

Estimation of local density

- $w_{sn} \equiv D |(c_n^s)_1|^2$, weight of each Lanczos eigenvalue.
- The local density is

$$\begin{aligned}\rho_s(M^2) &\simeq \frac{D}{4\pi P^+} \int e^{iM^2 x^+ / 2P^+} \sum_n |(c_n^s)_1|^2 e^{-iM_{sn}^2 x^+ / 2P^+} dx^+ \\ &\simeq \frac{D}{4\pi P^+} \sum_n |(c_n^s)_1|^2 2\pi \delta(M^2 / 2P^+ - M_{sn}^2 / 2P^+) \\ &\simeq \sum_n w_{sn} \delta(M^2 - M_{sn}^2),\end{aligned}$$

- Only the extreme Lanczos eigenvalues are good approximations to eigenvalues of the original P^- .
- The other Lanczos eigenvalues provide a smeared representation of the full spectrum.

Cumulative distribution function

- The contribution to the cumulative distribution function is

$$N_s(M^2) \equiv \int^M dM^2 \rho(M^2) \simeq \sum_n w_{sn} \theta(M^2 - M_{sn}^2).$$

- The full CDF is then approximated by the average

$$N(M^2) \simeq \frac{1}{S} \sum_s N_s(M^2).$$

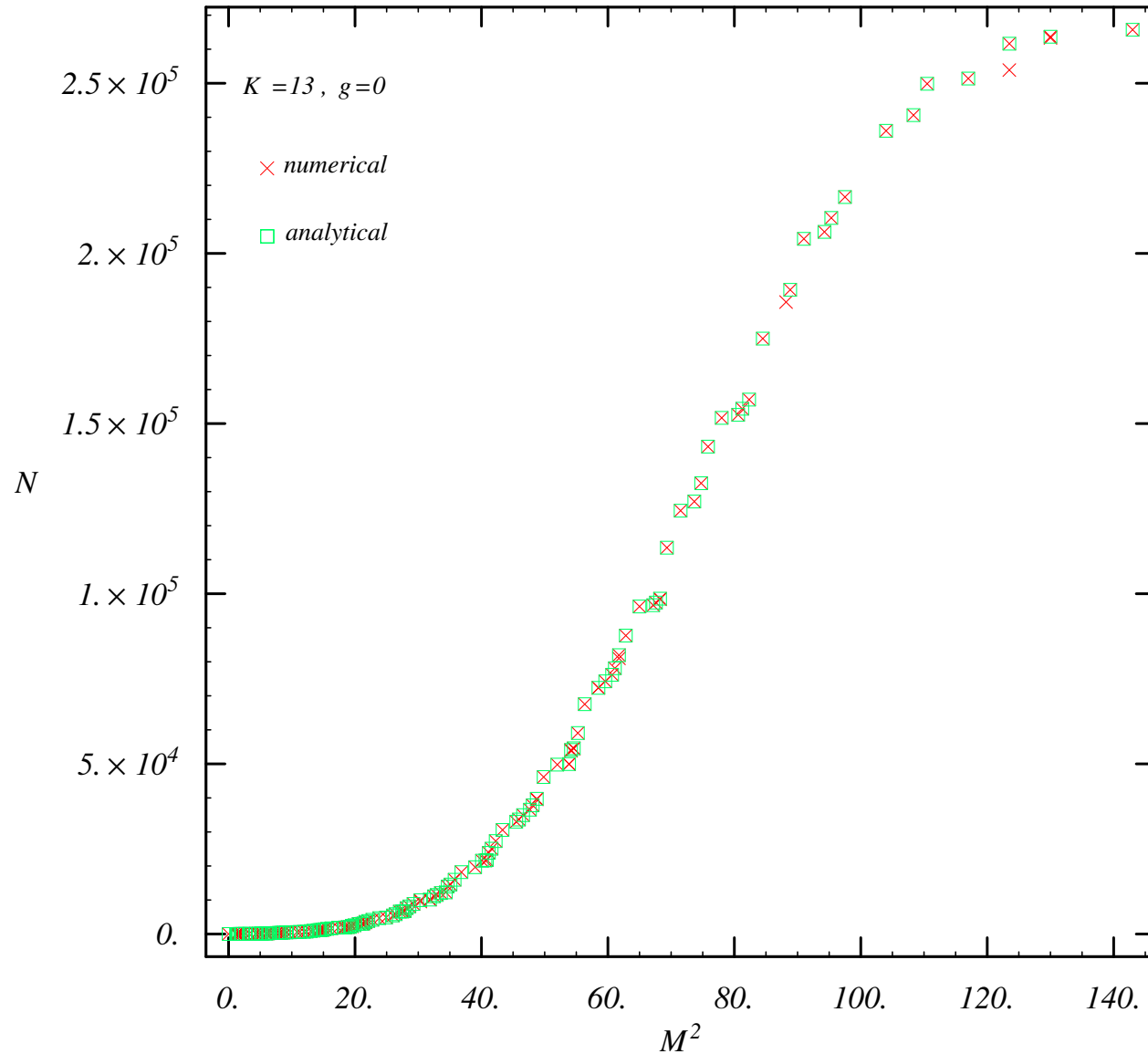
Averaging theta functions

- Use first sample run as a template for values M_{1n}^2 at which to evaluate N .
- The contributions of the other samples to N are estimated at these values by linear interpolation in cases where the Lanczos eigenvalues M_{sn}^2 are not the same as those in the first set.
- In cases where duplicate eigenvalues are generated by the Lanczos iterations, only one is included in the template and the associated weights are added together.

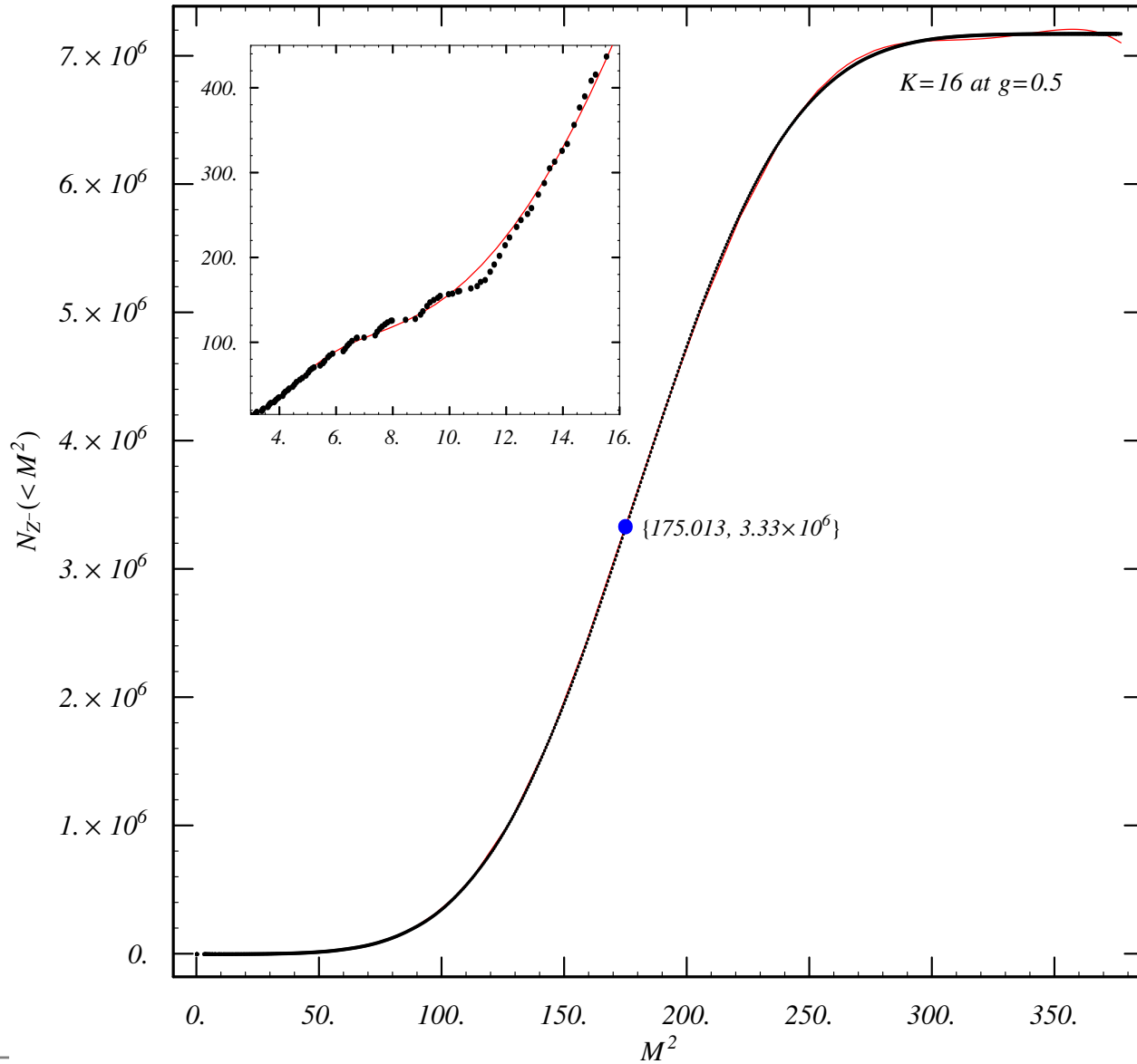
Convergence

- The convergence of the approximation is dependent on the number of Lanczos iterations per sample, as well as the number S of samples.
- Test runs indicate that the recommended value of 20 samples is sufficient.
- The number of Lanczos iterations needs to be on the order of 1000 per sample; using only 100 leaves errors on the order of 1-2%.

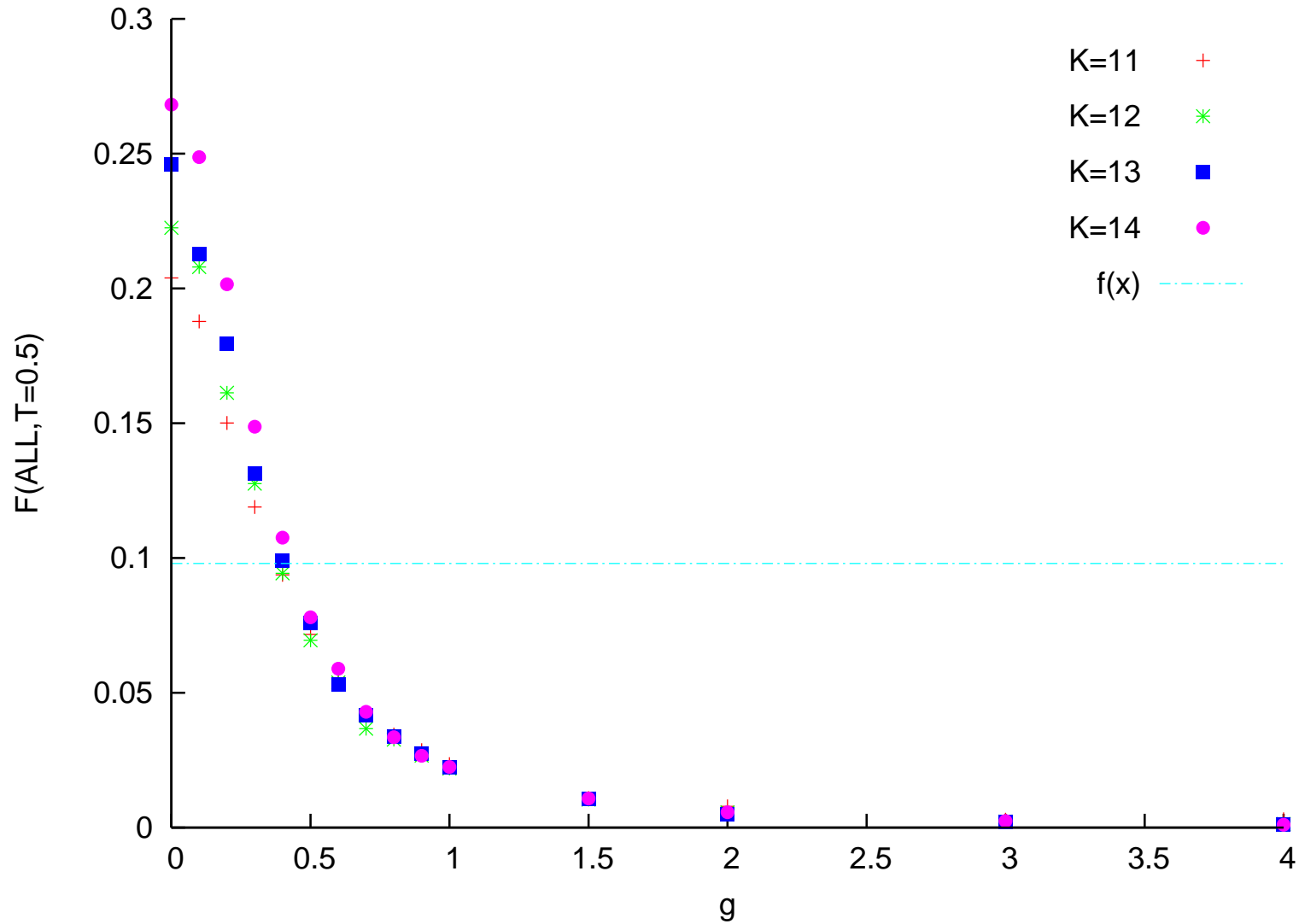
Cumulative distribution function



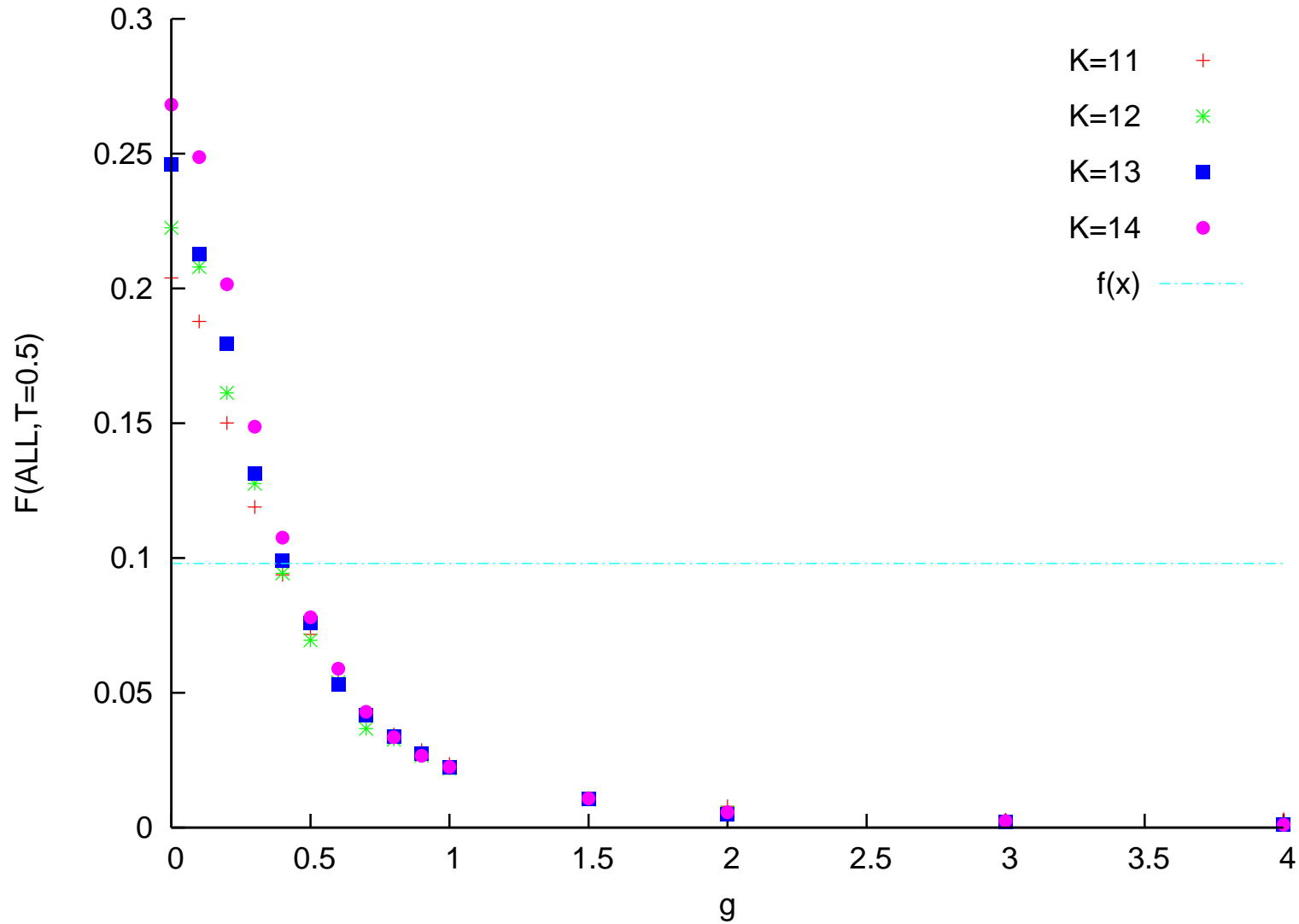
Cumulative distribution function



Free energy at fixed T



Free energy at fixed g



Future work

- SYM: higher resolution.
- SQCD:
 - 2+1
 - finite- N_c effects,
eg meson/glueball mixing, baryon states.



QCD