Winding String Dynamics **in** Time-Dependent β Deformed Background

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based on the work of [hep-th/0608136], to appear in PTP

Basic idea: deal with a solvable non-static string bg

Candidate

String theory on a *Lorentzian* flux-brane like bg

T-s-T transformation

A free CFT

+

twisted periodicity condition on closed strings

Exactly solvable to all order in α'

T-s-T: T-duality + shift of polar angle + T-duality

Quantize this free CFT to gain the exact string spectrum

L.Cornalba, M.Costa[hep-th/0203031]
N.Nekrasov[hep-th/0203112]

↓

Computable torus and cylinder amplitude of *Lorentzian* twisted string

Y.Hikida et al.[hep-th/0508003]

? ↓ ← We develop a way to check

Whether these encode closed string dynamics in time-dependent flux-brane like bg

A mini-superspace approach:

Capture
closed string dynamics
at zero-slope limit α' —> 0

i.e. point-particle approximation

Closed string Hamiltonian $L_0 + \tilde{L}_0$

Effective Laplacian

$$\triangle_0 = \frac{1}{e^{-2\Phi}\sqrt{-\det G}}\partial_\mu\left(e^{-2\Phi}\sqrt{-\det G}\;G^{\mu\nu}\partial_\nu\right)$$

Virasoro constraint ⇒ Wave equation

The curved bg under consideration will ...

Free rep with twisted b.c. via TsT

S.Forolov[hep-th/0503021]
J.Russo[hep-th/0508125]

$$X^{\pm}(\tau,\sigma+2\pi)=e^{\pm 2\pi\nu}X^{\pm}(\tau,\sigma)\ ,\ \ \nu=\beta(J_L+J_R)\ ,$$

$$X(\tau,\sigma+2\pi)=e^{-2\pi i\mu}X(\tau,\sigma)\ ,\ \ \mu=-\beta(\mathcal{J}_L+\mathcal{J}_R)$$

$\mathcal{J}_L+\mathcal{J}_R\ (J_L+J_R)$ is the boost (rotation) generator

On-shell constraint for the string state

$L_0+\tilde{L}_0$ =0

2nd order ODE

$$\ddot{\psi}(t)+\omega(t)^2\psi(t)=0$$

By using the mini-superspace method, we can ...

Goal(i):

Solve
the above time-dependent
ODE to obtain the Bogoliubov coeffecient

Extract
the string production rate,
then re-express it as an imaginary
part of a 1-loop vacuum free energy Z

Compare
Z with the CFT torus amplitude
(quasi-zero mode part) at zero-slope limit

Goal(ii):

Use
the wave function and DBI action

Extract
the disk one-point correlator $<O>$

Compare
$<O>$ with the cylinder amplitude from overlapping boundary sates at $\alpha' << 1$

Plan of the talk

1. Introduction
2. Bg description
3. Classical and CFT spectrum match
4. Classical computation
5. CFT computation
6. Summary

2. Background description

What is beta deformed bg? ***O. Lunin, J. Maldacena[hep-th/0502096]***

For any string bg with U(1) x U(1)

realized geometrically, i.e. a two torus,

do an SL(2,R) transformation to the parameter

$$\tau = B_{21} + i\sqrt{g} \;\rightarrow\; \tau' = \frac{\tau}{1 + b\tau}$$

such that $ds^2 = dr_1^2 + dr_2^2 + r_1^2 d\varphi_1^2 + r_2^2 d\varphi_2^2,$

$$ds^2 = dr_1^2 + dr_2^2 + \frac{r_1^2}{1 + b^2 r_1^2 r_2^2} d\varphi_1^2 + \frac{r_2^2}{1 + b^2 r_1^2 r_2^2} d\varphi_2^2 \,,$$

$$B_{\varphi_1 \varphi_2} = \frac{-b r_1^2 r_2^2}{1 + b^2 r_1^2 r_2^2} \,, \qquad e^{2(\Phi - \Phi_0)} = \frac{1}{1 + b^2 r_1^2 r_2^2}$$

By Wick rotation $r_1 \to it,\ \varphi_1 \to i\theta,\ b \to ib,$

$$ds^2 = -dt^2 + dr^2 + \frac{t^2}{1+b^2t^2r^2}d\theta^2 + \frac{r^2}{1+b^2t^2r^2}d\varphi^2$$

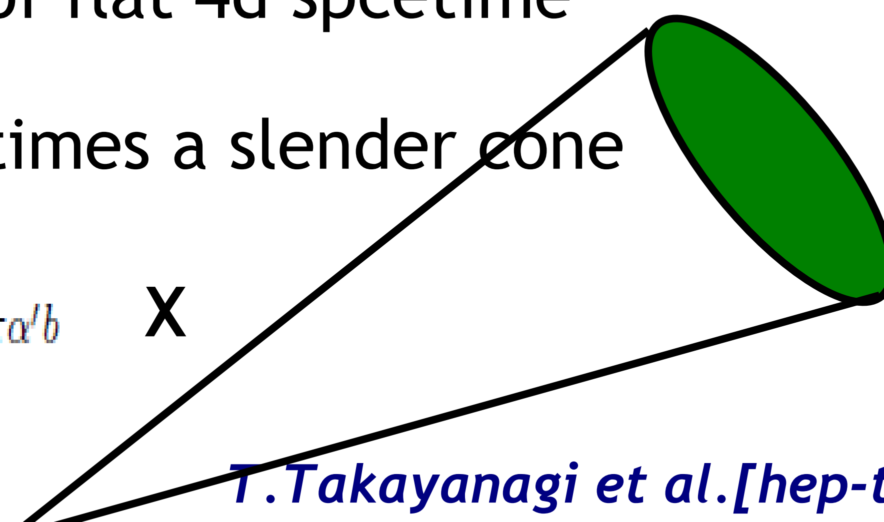
$$B_{\theta\varphi} = \frac{-bt^2r^2}{1+b^2t^2r^2}\ , \qquad e^{2(\Phi-\Phi_0)} = \frac{1}{1+b^2t^2r^2}$$

Crucial time-dependence

t=0, Milne wedge of flat 4d spcetime

t=infinity, Misner times a slender cone

$[\mathbb{R}^{1,1}/\mathbb{Z}]_{\Delta=2\pi\alpha' b}$ x



T.Takayanagi et al.[hep-th/0509036]

3. Classical and CFT spectrum match

Ansatz of wave function (embedded in 26d bosonic string):

$$\Psi(t,r,\theta,\varphi,\vec{x}) = \Psi_t(t)\Psi_r(r)e^{ik\theta+im\varphi+i\vec{p}\vec{x}} \,, \quad m \in \mathbb{Z} \,, \quad k \in \mathbb{R}$$

Temporal: $$\Big[-\frac{1}{t}\frac{\partial}{\partial t}t\frac{\partial}{\partial t} - \frac{k^2}{t^2} - b^2m^2t^2\Big]\Psi_t(t) = E^2\Psi_t(t) \,,$$

Radial: $$\Big[-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \big(\frac{m^2}{r^2} + b^2k^2r^2\big)\Big]\Psi_r(r) = R^2\Psi_r(r) \,.$$

$\Psi_r(r) = Cr^{|m|}e^{\frac{-b|k|r^2}{2}}L_j^{|m|}(b|k|r^2)$ ***L*(x) is a Laguerre polynomial**

$E^2 - \vec{p}_{22}^{\,2} = R^2 = 2b|k|\big(2j + |m| + 1\big)\,, \quad j:$ non-negative integer

Dealing with the temporal part

$$t \to e^{\xi},\ \Psi_t(t) \to Y(\xi)\ (0 \le t \le \infty),$$

$$\left(\frac{d^2}{d\xi^2} + E^2 e^{2\xi} + b^2 m^2 e^{4\xi} + k^2\right) Y(\xi) = 0$$

By $W(z) = e^{\xi} Y(\xi)$,

where $z = -ib|m|e^{2\xi}$ is an imaginary variable

$$\left[\frac{d^2}{dz^2} + \left(\frac{-1}{4} + \frac{\lambda}{z} + \frac{1-4\eta^2}{4z^2}\right)\right] W_{\lambda,\eta}(z) = 0\,, \qquad \lambda = \frac{i}{4b|m|} E^2\,, \qquad \eta = \frac{i|k|}{2}$$

This is the standard Whittaker equation

Compare the spectrum:

$$E^2 - \vec{p}_{22}^{\,2} = R^2 = 2b|k|\Big(2j + |m| + 1\Big)$$

CFT side:

$$\frac{\alpha'}{2}M^2 := \frac{1}{2}\Big(\alpha_0^+\alpha_0^- + \alpha_0^-\alpha_0^+ + \tilde{\alpha}_0^+\tilde{\alpha}_0^- + \tilde{\alpha}_0^-\tilde{\alpha}_0^+\Big) - \frac{\alpha'}{2}\vec{p}_{22}^{\,2}$$

after $\quad \alpha' \to 0\ , \quad \beta \to 0\ , \quad b = \frac{\beta}{\alpha'} = \text{fixed}$

$$M^2 = \frac{2\mu}{\alpha'}(l_L + l_R + 1) \qquad \mu = \alpha' b k$$

where $l_{L,R}$ are the quasi-zero mode's occupation #

Under identifying $\quad m = l_L - l_R\ , \qquad 2j + |m| \qquad = l_L + l_R$

We see that the spectra from two sides match

4. Classical computation

Calculating winding string production rate

time interval $0 \leq t \leq \infty$

Tool: Bogoliubov transformation

$$\psi_\sigma^{in} = \alpha_\sigma \psi_\sigma^{out} + \beta_\sigma \psi_{-\sigma}^{out*} \ ,$$

$$\psi_\sigma^{out} = \alpha_\sigma^* \psi_\sigma^{in} - \beta_\sigma \psi_{-\sigma}^{in*} \ .$$

Select *in* & *out* vacua

$$\psi_\sigma^{in} = |t|^{-1} M_{\lambda,-\mu}(-z)$$

$$\psi_\sigma^{out} = e^{-\xi} W_{\lambda,\eta}(-z) \text{ where } \sigma \text{ abbreviates } (k, m, \vec{p})$$

W(z), *M*(z): Whittaker and Kummer function are two linearly independent solution of Whittaker equation.

Asymptotics of the *in* & *out* states

t near 0, $\psi_\sigma^{in}(t) \propto e^{-i|k|\log|t|}$

t near infinity, $\psi_\sigma^{out}(t) \propto |t|^{2\lambda-1} e^{\frac{-ib|m|t^2}{2}}$

Both approach the positive frequency mode,

via Bogoliubov transformation → winding string production

Using an identity between the Kummer and Whittaker function

$$W_{\lambda,\eta}(-z) = \frac{\Gamma(-2\eta)}{\Gamma(\frac{1}{2}-\eta-\lambda)} e^{i\pi(\eta+\frac{1}{2})}[M_{\lambda,-\eta}(-z)]^* + \frac{\Gamma(2\eta)}{\Gamma(\frac{1}{2}+\eta-\lambda)} M_{\lambda,-\eta}(-z)$$

We can read off the Bogoliubov coefficients α and β, such that the production rate is

$$\left|\frac{-\beta_\sigma}{\alpha_\sigma^*}\right|^2 = |\gamma_\sigma|^2 = e^{2\pi i\eta}\left|\frac{\Gamma(\frac{1}{2}-\eta+\lambda)}{\Gamma(\frac{1}{2}-\eta-\lambda)}\right|^2 = \frac{1+e^{\pi\left(|k|-\frac{E^2}{2b|m|}\right)} e^{-2\pi|k|}}{1+e^{\pi\left(|k|-\frac{E^2}{2b|m|}\right)}}$$

Rewrite the production rate as

$$e^{\sum_{\sigma,j} \log|\gamma_\sigma|^2} = e^{-(\mathcal{A}-\mathcal{B})}$$

$$\mathcal{A} = \sum_{\sigma,j} \log\left[1 + e^{\pi(|k| - \frac{E^2}{2b|m|})}\right]$$

$$= \sum_{\sigma,j}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{\frac{-n\pi\alpha' E'^2}{2|\nu|}} \quad = -2\mathrm{Im}\left[\int_0^\infty \frac{du}{u} \mathrm{Tr}\, e^{-\pi u \mathcal{H}}\right]$$

An imaginary part of a 1-loop vacuum free energy

$$\mathrm{Tr} e^{-\pi u \mathcal{H}} = \sum_{k,m,j} \int_{-\infty}^{\infty} dE'^2 \rho_{k,m,j}(E'^2) e^{-\frac{1}{2}\pi u \alpha' \left[E'^2 + 2b|k|(2j+1)\right]}$$

$$= \sum_{k,m} \frac{1}{4i\sin(\pi|\nu|u)\ \sin(-\pi|\mu|iu)}$$

$E'^2 = E^2 - 2b|k||m|$ & $\vec{p}_{22}^{\,2}$ is neglected for simplicity

We have used the following formulas among the derivation:

$$\frac{1}{2i\sin y} = \sum_{j=0}^{\infty} e^{-i(2j+1)y}$$

$$\rho_{k,m,j}(E'^2) = \frac{i}{2b|m|}\log\Lambda + \frac{1}{2\pi}\frac{d}{dE'^2}\log\frac{\Gamma(\frac{1}{2} - i\frac{E'^2}{4b|m|})}{\Gamma(\frac{1}{2} + i\frac{E'^2}{4b|m|})}$$

A parallel can be drawn between this case and charged particles moving in the Rindler space $ds^2 = dr^2 - r^2 d\chi^2$,

A: induced by bg B-field ⟺ usual Schwinger effect

B: Milne metric (b=0) ⟺ gravitational Unruh effect
since B-->production rate of
untwisted string

Disk one-point correlator:

Expand w.r.t. scalar dilaton fluctuation

$$S^{DBI} = S_0 + \int d^{p+1}x \, \frac{\delta S}{\delta \Phi}\bigg|_0 \delta\Phi + ...$$

Identifying the fluctuation with the wave function

$$\langle \Psi \rangle_{disk} = \tau_p \int d^{p+1}x \, e^{-\Phi} \sqrt{-\det(G+B)}\Psi(x)$$

Since D-brane wrapping the whole geometry carries no twisted sector, we use the following two kinds of D-brane, i.e.

(i) D-brane instaton wrapping only twisted X:

$$\int d\varphi dr r \, C\Psi_r(r) = \sqrt{\frac{b|k|}{\pi}} \int_0^\infty dx \, L_j^0(x) e^{-x/2} = 2(-)^j \sqrt{\frac{\pi}{b|k|}} \qquad C = \sqrt{\frac{(b|k|)^{|m|+1} j!}{\pi(j+|m|)!}}$$

(ii) D1-brane wrapping light-cone directons:

There remains normalization ambiguity though

5. CFT computation

$$L_0 = -1 + \frac{\nu^2}{2} - \frac{\hat{\mu}^2}{2} + \nu \mathcal{J}_L + \hat{\mu} J_L + N + \frac{\alpha'}{4}\vec{p}_{22}^2$$

Torus amplitude: $Z(\tau) = \mathrm{Tr}\, q^{L_0}\bar{q}^{\tilde{L}_0}$,

$$\tilde{L}_0 = -1 + \frac{\nu^2}{2} - \frac{\hat{\mu}^2}{2} - \nu \mathcal{J}_R - \hat{\mu} J_R + \tilde{N} + \frac{\alpha'}{4}\vec{p}_{22}^2$$

$$Z(\tau) = \frac{V_{22}}{(2\pi)^{22}(\alpha'\tau_2)^{11}} \int d^2\chi_r d^2\chi_b \, \frac{1}{\left|\vartheta_1(i\chi_b|\tau)\vartheta_1(\chi_r|\tau)\eta(\tau)^{18}\right|^2}$$

$$\frac{1}{(2\beta\tau_2)^2} \exp\left[\frac{\pi}{\beta\tau_2}(\chi_b\bar{\chi}_r - \chi_r\bar{\chi}_b)\right]$$

$$\times \exp\left[\pi\frac{(\chi_b - \bar{\chi}_b)^2 - (\chi_r - \bar{\chi}_r)^2}{2\tau_2}\right]$$

point particle approximation

$$\alpha' \ll 1, \quad \tau_1 = 0,\ \tau_2 = u \qquad \chi_b \to -\nu\tau,\ \chi_r \to -\hat{\mu}\tau$$

The (anti-)holomorphic part of quasi-zero mode is summarized as

Agreement

$$\frac{1}{4\sin(\pi\nu u)\ \sin(-\pi\mu i u)}$$

Cylinder amplitude:

D-brane instanton wrapping twisted X

$$\frac{\alpha'\pi}{2}N_1'^2\int_0^\infty ds\,\langle\langle Ins;k,B|e^{-\pi s(L_0+\bar{L}_0)}|B,k;Ins\rangle\rangle =$$

$$\int_0^\infty ds\frac{(\frac{N_1'}{\mathcal{N}_1})^2 4\pi^2\alpha' e^{\pi\beta^2k^2s-\frac{\pi k^2\alpha'}{2t_0^2}s}}{s^{\frac{23}{2}}(8\pi^2\alpha')^{\frac{3}{2}}\vartheta_1(-i\beta|k|s\,|\,is)\eta(is)^{21}}$$

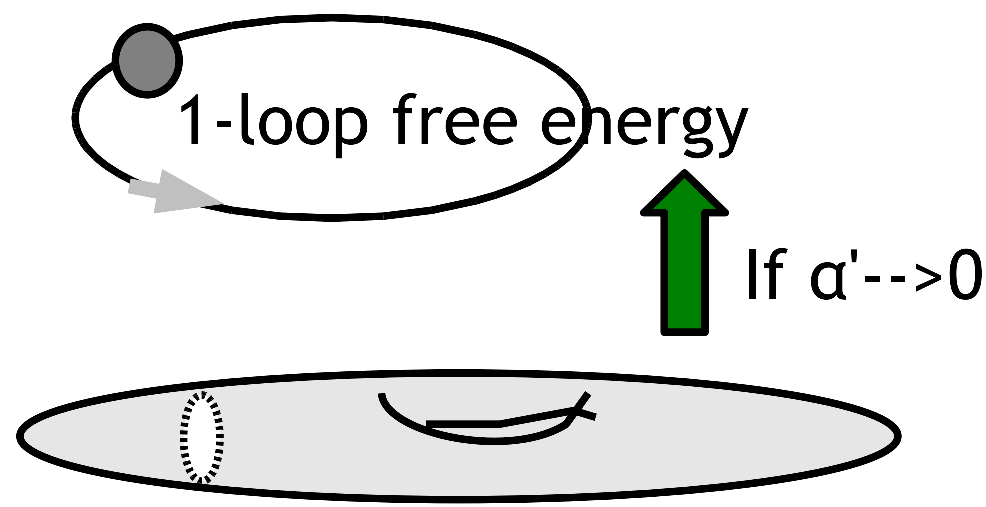
$(\frac{N_1'}{\mathcal{N}_1})^2$ ⟹ ($\alpha' \ll 1$) **<*O*>** **Agreement**

$\mathcal{N}_1$: usual D1-brane normaliation

Summary:

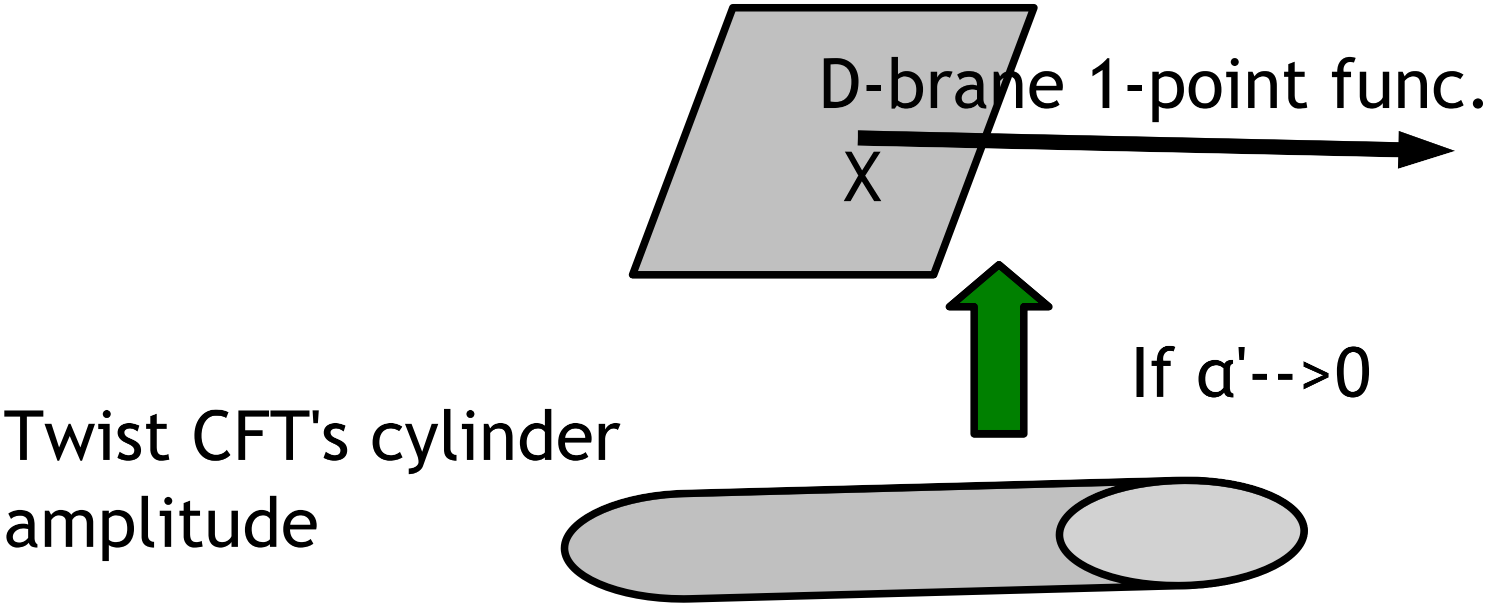
1. By the wave mechanical approach in flux-brane like bg ⇒ β-deformed bg, we can understand the twist CFT's quasi-zero mode part amplitude

2. Via the Bogoliubov transformation & the optical theorem, we extract 1-loop free energy



Twist CFT's torus amplitude

3. Via DBI action and wave function, we extract disk 1-point function



Outlook:
More complicated twisted bg, e.g. 3-parameter model, etc.

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