Recent results from Domain Wall Fermions in Full QCD

Chris Dawson

[RBC-UKQCD Collaboration]
Introduction

Plan:

- Domain Wall Fermions
- (RBC-UKQCD) 2+1 flavour QCD simulations
- Two applications in kaon physics:
  - Kaon Beta decay
  - Kaon B-parameter

Rationale for Domain Wall fermions

- Traditional lattice actions break either flavour (Staggered) or chiral (Wilson) symmetry at finite lattices spacing.
- **Domain Wall Fermions**: Exact Flavour symmetry, greatly suppressed breaking of Chiral symmetry.
- While more expensive, the hope is that the symmetry properties at finite lattice spacing outweigh the cost (lattice simulations $\propto 1/a^8$)
- this is a quantity dependent question.
Domain Wall Fermions

- Action in 5th dimension asymmetric w.r.t. chirality.
- Define 4d quark fields on the wall
- Couple the two walls with a mass term $m_f \bar{q} q$

- For finite $L_S$ chiral symmetry is broken, leading to an additive shift of the mass
  
  $$m_f \rightarrow m_f + m_{res}$$

- $m_{res} \rightarrow 0$ as $L_S \rightarrow \infty$; The cost in computer time $\propto L_S$
  - Can get small $m_{res}$ (few MeV) for reasonable $L_S$ ($O(10)$).
Dynamical DWF on the QCDOC

- DWF work well in the quenched and two-flavour theory.
- Vital to move to full QCD

The QCDOC computers: RBRC (BNL), UKQCD (Edinburgh), US Machine (BNL).
- each $\sim 10$ TFlops (peak).

Joint project UKQCD, RBC using (parts of) all three machines: 2+1 flavour Dynamical DWF
- “light” dynamical masses ($\rightarrow m_s/5$)
- (at least) two lattice spacings...
- (at least) two volumes
# RBC-UQCD DWF Collaboration (probably a little out of date)

## RBRC
- Blum, Tom (U Conn)
- Dawson, Chris
- Hashimoto, Koichi
- Izubuchi, Taku (Kanazawa)
- Ohta, Shigemi (KEK)
- Sasaki, Shoichi (KEK)
- Yamazaki, Takeshi

## BNL
- Jung, Chulwoo
- Sholtz, Enno
- Soni, Amarjit

## Columbia
- Aubin, Christopher
- Christ, Norman
- Cohen, Saul
- Li, Sam
- Lin, Meifeng
- Lin, Huey Wen
- Loktik, Oleg
- Mawhinney, Robert

## UKQCD
- Allton, C.
- Antonio, D. J.
- Bowler, K. C.
- Boyle, P. A.
- Clark, M. A.
- Jüttner, A.
- Joo, B.
- Kennedy, A. D.
- Kenway, R. D.
- Noaki, J.
- Maynard, C. M.
- Pendleton, B. J.
- Sachrajda, C.
- Traveni, A.
- Tweedie, R. J.
- Yamaguchi, A.
- Zanotti, J.
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<th>( L * a ) (Fm)</th>
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- Have completed \( 16^3 \times 32 \) run (at a little coarser lattice spacing, and lighter masses), and are in the process of completing \( 24^3 \times 64 \) run.

- Scale setting runs for the small lattice spacing \( 32^3 \times 64 \) lattices currently underway.
Some results

In the remainder of this talk, I will cover two quantities in Kaon Physics

Kaon Beta Decay.

The Kaon B-parameter.

this is an on-going calculation, so all results are preliminary

Basic Parameters:

single lattice spacing : \( a^{-1} \sim 1.6 \text{ GeV} \)

\( m_{\text{res}} \sim 5 \text{ MeV} \)

spatial extent \( \sim 2 \text{ fm} \) and \( \sim 3 \text{ fm} \) for \( 16^3 \times 32 \) and \( 24^3 \times 32 \) respectively.

three dynamical masses, lightest \( \sim m_s/3 \)
Kaon Beta decay

\[ K^0 \rightarrow \pi^- L^+ \nu_l; \ K^+ \rightarrow \pi^0 L^+ \nu_l \]

where \( l \in \{e, \mu\} \).

- Integrating out the Weak force (and heavy quarks): \( \Gamma_{K_{l3}} \propto |V_{us}|^2 |f_+(0)|^2 \)

\[
\langle \pi(p_f) | \bar{s} \gamma_\mu u | K(p_i) \rangle = (p_i + p_f)_\mu f_+(q) + q_\mu f_-(q) ; \ q = p_i - p_f
\]

where the LHS is evaluated in (3 flavour) QCD.

- Used to extract \( |V_{us}| \).
  - \( f_K/f_\pi \) competitive (more developed calculation on the lattice)

- Why is this process so nice?
  - Insertion of vector current: favourable symmetry properties (no renormalisation in the continuum).
  - Don’t need much non-perturbative input...
Calculating $f_+(0)$

- Ademollo-Gatto theorem: $f_+(0) = 1 - O((m_s - m_u)^2)$

Expand the form factors in Chiral Perturbation Theory:

$$f_+(q^2) = 1 + f_2 + f_4 + \cdots$$

with $f_i$ of $O(M_i^i/f_i)$ in ChiPT

- $f_2$:
  - $f_2$ depends on no new low energy constants. Can be worked out from $M_K$, $M_\pi$ and $f_\pi$.
  - $-0.023$ using values from experiment.

- $f_4$:
  - Calculated in Chiral Perturbation Theory by Bijnens and Talavera. In *principle*, can be constrained by the experimentally measured slope of $f_0(q^2)$, but needs better experimental resolution.
  - $-0.016(8)$ from quark model [Leutwyler and Roos, 1984]
The double ratio method

Rather than work with the three-point function of interest directly, the double ratio is used. ([Becirevic et al, hep-lat/0403217] c.f. [Hashimoto et al, 2000]).

\[
\frac{\langle \pi | \bar{s} \gamma_0 u | K \rangle \langle K | \bar{u} \gamma_0 s | \pi \rangle}{\langle \pi | \bar{u} \gamma_0 u | \pi \rangle \langle K | \bar{s} \gamma_0 s | K \rangle} = \left[ f_0 \left( q_{max}^2 \right) \right]^2 \frac{(M_K + M_\pi)^2}{4M_K M_\pi} ; \quad q_{max}^2 = (M_K - M_\pi)^2
\]

\[
f_0(q^2) = f_+(q^2) + \frac{q^2}{M_K^2 - M_\pi^2} f_-(q^2)
\]

Scalar form factor:

\[
f_0(0, M_\pi, M_K) = f_+(0, M_\pi, M_K)
\]

This approach has several advantages:

- Small statistical error ( < 0.1% )
- Exactly unity, and exactly \( f_+(0) \), on the lattice in \( SU(3) \) limit.
The double ratio method...

This just gives $f_0(q^2, M_\pi, M_K)$. Need to

1. Extrapolate, in $q^2$, to $f_0(0) = f_+(0)$ at a fixed (non-physical) mass.
2. Extrapolate to physical masses.

For 1) lattice data with explicit insertion of momenta is needed.

▷ [Becirevic et al, hep-lat/0403217] show how to use various ratios to allow an extraction with a small enough error-bar to be useful.
Double Ratio/Momentum Extrapolation \((N_f = 2 + 1)\)

- From James Zanotti’s talk at lattice 2006
- Zero momentum injection (small momentum transfer as lattice pion and lattice kaon are close in mass).
- Several mass differences shown

\[ f_0(q^2) = f_0(0)/(1 - \lambda^{(pol)} q^2) \]
\[ f_0(q^2) = f_0(0)(1 + \lambda^{(1)} q^2) \]
\[ f_0(q^2) = f_0(0)(1 + \lambda^{(2)} q^2 + c q^4) \]

- Requires some ansatz to fit to:
- Larger lattice has smaller minimum lattice momenta
Mass extrapolation ($N_f = 2 + 1$)

- Unitary points shown.
- $\Delta f \propto (m_s - m_d)^2$: Ademollo-Gatto
- Higher order terms, so looks at:

$$R = \frac{\Delta F}{m_K^2 - m_{\pi}^2}$$

and try

$$R = a + b \left( m_K^2 + m_{\pi}^2 \right)$$
Dramatically smaller error-bar: Larger volume certainly helping

However:

- Chiral extrapolation for all measurements is over a large range.
- Ansatz used for momentum extrapolation. (systematics not yet studied)
- Lattice spacing, Volume effects?
- Low statistics on larger volume!

For “fun”, using $|V_{us} f_+(0)| = 0.2169(9)$:

$$|V_{us}| = 0.2241(9)_{\exp(4)} f_+(0)$$

and

$$1 - |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 0.0015(7)$$

versus $0.0008(11)$ (PDG)

Encouraging preliminary result. But it is preliminary, and not including all systematics...
$B_K$ is the low energy matrix element relevant to CP-violation in $K^0 - \bar{K}^0$ mixing.

Integrate all the particles with masses $\gg \Lambda_{QCD}$:

$$|\epsilon| = C_\epsilon A^2 \chi^6 \bar{\eta} \left[ -\eta_1 S(x_c) + \eta_2 S(x_t) A^2 \chi^4 (1 - \rho) + \eta_3 S(x_c, x_t) \right] \hat{B}_K$$

Need to calculate this on the lattice:

$$B_K = \frac{\langle \bar{K}^0 | O_{LL} | K^0 \rangle}{\frac{8}{3} m_K^2 f_K^2}$$

renormalised in some scheme at some scale.
Operator Mixing and $B_K$

In the continuum there is one operator that contributes to $B_K$. It is of the form:

$$O_\Gamma = \bar{s} \Gamma i d \, \bar{s} \Gamma i d$$

with the gamma structure:

$$VV + AA \equiv \gamma_\mu \otimes \gamma_\mu + \gamma_\mu \gamma_5 \otimes \gamma_\mu \gamma_5 ;$$

which is simply the parity conserving part of

$$(V - A) \otimes (V - A)$$

- **Staggered fermions**: extra flavours ("tastes" - taste mixing problem)
  - Many other operators can mix.
  - Resolve using 1-loop perturbation theory.
- **Wilson fermions**: Broken Chiral Symmetry.
If chiral symmetry is broken, four other operators can mix (the four other possible gamma matrix structures)

$$\langle \bar{K}^0 | O_{VV+AA} | K^0 \rangle_{\text{latt}} \propto \langle \bar{K}^0 | O_{VV+AA} | K^0 \rangle_{\text{ren}} + \sum_{i \geq 2} c_i \langle \bar{K}^0 | O_{MIX,i} | K^0 \rangle_{\text{ren}}$$

These operators, of course, have a different chiral structure.

Mixing is hard to control using perturbation theory; First order chiral perturbation theory predicts that

$$\langle \bar{K}^0 | O_{VV+AA} | K^0 \rangle \propto M_k^2$$

and,

$$\langle \bar{K}^0 | O_{\text{THE REST}} | K^0 \rangle \propto \text{constant}$$

small enough mass, wrong chirality operators will dominate.
Domain Wall Fermions

 Domain Wall Fermions "almost" preserve chiral symmetry.

 "almost" no mixing with the wrong chirality operators

 wrong chirality matrix elements $O(10)$ times larger than signal at masses of interest.

 How much chiral symmetry is enough?

 Simple model:

 One trip through the bulk: suppression factor of $O(am_{\text{res}})$

 Operator of interest is $(V - A)^2$: four left-handed fields.

 wrong chirality operators: two left-handed, two right-handed

 $\rightarrow O((am_{\text{res}})^2) \sim 10^{-6}$
Approach

I’m going to show some RBC results in the quenched approximation (Jun Noaki) from two different lattice spacings.

- $a^{-1} = 2 \text{ GeV} \ (L_s = 16)$ and $3 \text{ GeV} \ (L_s = 10)$
- $\sim 1.5 \text{fm}^3 \times 3 \text{fm} \ \text{box}$
- degenerate masses

- Put operator with quantum numbers for Kaon at timeslices 4 and 28 (2 GeV)
- Move effective Weak vertex over all timeslices
  - Plateau should appear for large separation.
Quenched Domain wall fermions results for the ratio used to extract $B_K$. 
Chiral Fits and Extraction of $B_K$

Predicted NLO ChPT:

$$B_K = b_0 \left( 1 - \frac{6}{(4\pi f)^2}M_K^2 \ln \left[ \frac{M_K^2}{(4\pi f)^2} \right] \right) + b_1 M_K^2$$

DWF(RBC) $a^{-1} = 2\text{GeV}$

DWF(RBC) $a^{-1} = 3\text{GeV}$
Extrapolate to the continuum as

\[ A + B a^2 \]

(would be linear in \( a \) without chiral symmetry).

Continuum limit consistent with CP-PACs Iwasaki/DWF calculation using perturbative renormalization.
Quenched World Average

Treat all the errors as statistical: all (continuum extrapolated) combined give

\[ B_{K}^{NDR}(2\text{GeV}) = 0.587(13) \]

All \( a^2 \) extrapolated, published gives

\[ B_{K}^{NDR}(2\text{GeV}) = 0.582(17) \]
- both with good \( \chi^2/\text{dof} \)

Errors not all statistical

\[ B_{K}^{NDR}(2\text{GeV}) = 0.58(3) \]

c.f \( B_{K}^{NDR}(2\text{GeV}) = 0.58(4) \) [Shoji Hashimoto (ICHEP 2004)]
$B_K$ and the CKM

Recall:

$$|\epsilon_K| = C\epsilon A^2 \lambda^6 \eta \left[ -\eta_1 S(x_c) + \eta_2 S(x_t) A^2 \lambda^4 (1 - \bar{\rho}) + \eta_3 S(x_c, x_t) \right] B_K$$

- $A, \lambda$ already well known ($\sim 5\%$).
- This is the CKMfitter group’s plot from EPS 2005.
- $B_K(\overline{MS}, 2 \text{ GeV}) = 0.58(3)(6)$

- The dominant error is from a (bad) guess of the quenching error.
- Need full QCD calculations, multiple lattice spacings, small masses...
\(N_f = 2\) Dynamical Domain Wall Fermions

- Repeat the same calculation for the two-flavour case
  - Now need chiral PT to perform an extrapolation! (with heavy masses)
  - Small downward trend with dynamical mass resolved within statistics
  - Big jump down (single lattice spacing)

Looks very dramatic, but...
$N_f = 2$ Dynamical Domain Wall Fermions

- “Suggestive” graph with the quenched and dynamical DWF results on, with the $a^2$ extrapolation on it.
- (maybe) not a very sensible thing to plot
- Our dynamical result is consistent than the quenched results closest in lattice spacing.
- This is the information used to estimate the systematic error due to quenching.

- need:
  - two lattice spacings
  - larger volumes
  - smaller masses
  - correct number of quarks
2+1 flavour Dynamical DWF; Small Volume

▷ Saul Cohens talk at lattice 2006.

▷ Example plateau extraction

▷ Chiral perturbation theory fit to the mass dependence (degenerate points)

▷ Doesn’t fit too well
  ▷ works well as an interpolation
  ▷ extrapolation questionable (masses too large)
Preliminary number: $B_K(M_S, 2GeV) = 0.546(10)(11)$

- first error statistics, second from the renormalisation factor (conservative).

Again, useful to look at a scaling plot:

Moving towards the Quenched Iwasaki data (which used perturbative renormalisation)

currently finalizing renormalisation factors, working on “error budget”
Larger Volume Work

Volume Comparison

$m_{\text{dyn}}=0.01,0.01$ ; $m_{\text{val}}=0.01,0.01$

▷ 22 configs for the large volume; 75 for small.
Dynamical Domain Wall Fermion simulations are well under way

I’ve shown preliminary result for two quantities

Kaon Beta Decay

relatively undeveloped quantity on the lattice
need further study of systematics

Kaon B-parameter

Good agreement in the quenched approximation between different approaches
DWF ideally suited for this calculation because of their good symmetry properties
Even with a single lattice spacing, useful information can be gained.

Larger volumes, smaller lattice spacings on the way...