Non-Abelian Vortices of Higher Winding Numbers

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work in collaboration with

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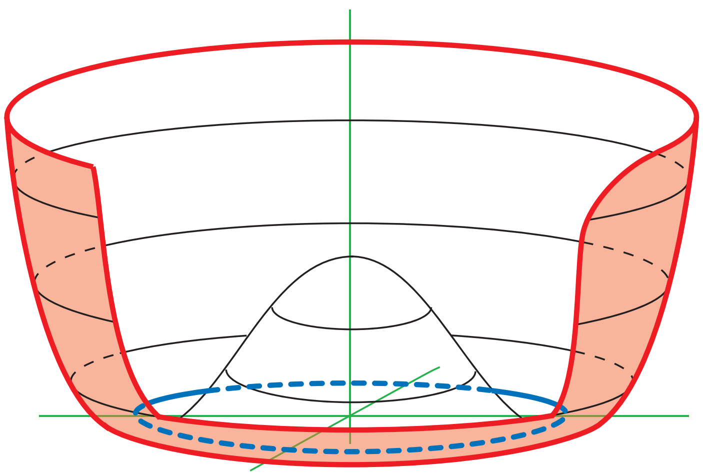
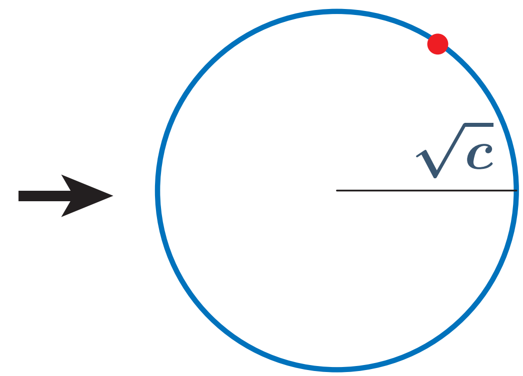
1. INTRODUCTION: Non-Abelian Vortex

Vortex is one of the important solitons in many areas of physics.
Especially, vortices may apper in the Abelian-Higgs model (‘70s Nielsen-Olesen) when the Abelian gauge symmetry is spontaneously broken.



scalar potential



vacuum manifold $(\mathcal{M} \simeq S^1)$

Meissner effect (usual superconductor):
Condensation of electrically charged particle $\Rightarrow$ Magnetic flux is squeezed

Recently, an extension of the Nielsen-Olesen vortex is found in $U(N)_c$ Yang-Mills theory coupled with N Higgs fields in fundamental representation:

$$\mathcal{L} = \mathrm{Tr}_c \left[-\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} - \mathcal{D}_\mu H \mathcal{D}^\mu H^\dagger - \lambda \left(c \mathbf{1}_N - H H^\dagger \right)^2 \right]$$

where the Higgs field H is $N_{(\text{color})} \times N_{(\text{flavor})}$ complex matrix.
We will choose critical (BPS) coupling $\lambda = g^2/4$ and set $c > 0$ for stable vortices.

This model has $SU(N)_f$ flavor symmetry besides $U(N)_c$ gauge symmetry.

The BPS equation of the non-Abelian vortices on the x^1, x^2 plane is given by

$$(\mathcal{D}_1 + i\mathcal{D}_2) H = 0, \quad F_{12} = -\frac{g^2}{2} \left(c \mathbf{1}_N - H H^\dagger \right).$$

Their solutions saturate the useual Bogomolnyi bound

$$\mathcal{E} \geq 2\pi k c \quad (k : \text{winding number}).$$

Usually, repulsive and attractive forces between multiple BPS solitons are completely canceled out, so they may have lots of moduli paramters, say positions of them. To study the structure of the moduli space of the solitons is basic and important work (ADHM for instanton / Nahm for monopole).

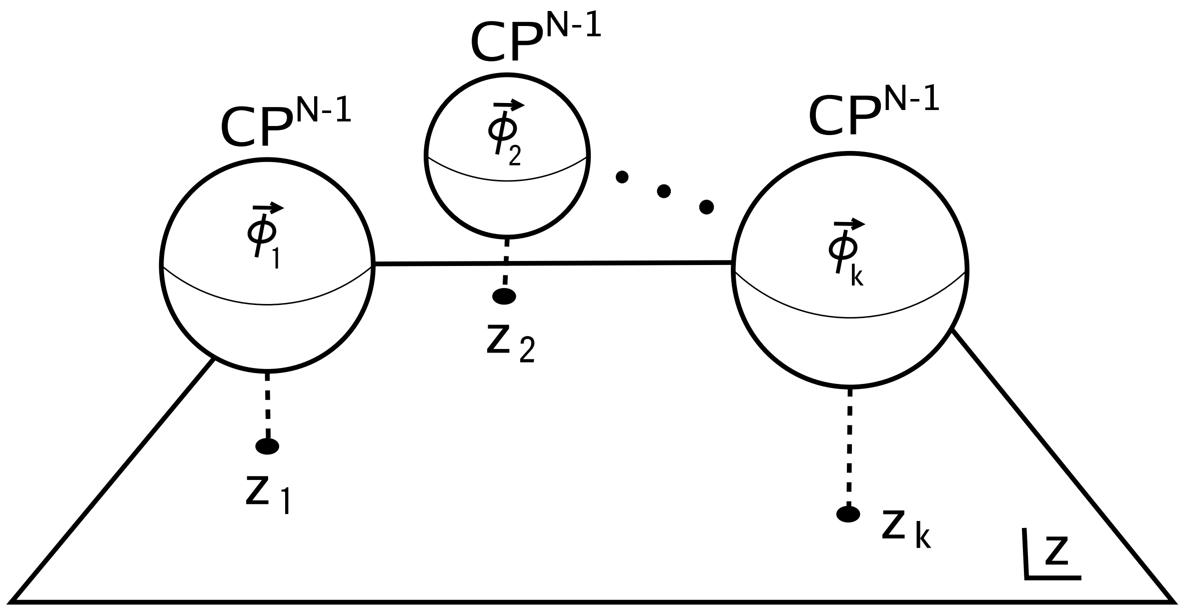
Nielsen-Olesen (Abelian) vortex:
There are many works for the moduli space of the Nielsen-Olesen vortices.

non-Abelian vortex:
Hanany-Tong proposed the moduli space of the non-Abelian vortices from the viewpoint of D-brane configurations in type IIA/B string theories.

$\Updownarrow$

We constructed complete moduli space of the non-Abelian vortices by using the moduli matrix formalism from purely field theoretical viewpoint.

It was found that minimal winding non-Abelian vortex in $U(N)$ YM has internal orientation $\mathbb{C}P^{N-1}$ besides translation $\mathbb{C}$: $\mathcal{M}_{k=1} = \mathbb{C} \times \mathbb{C}P^{N-1}$.

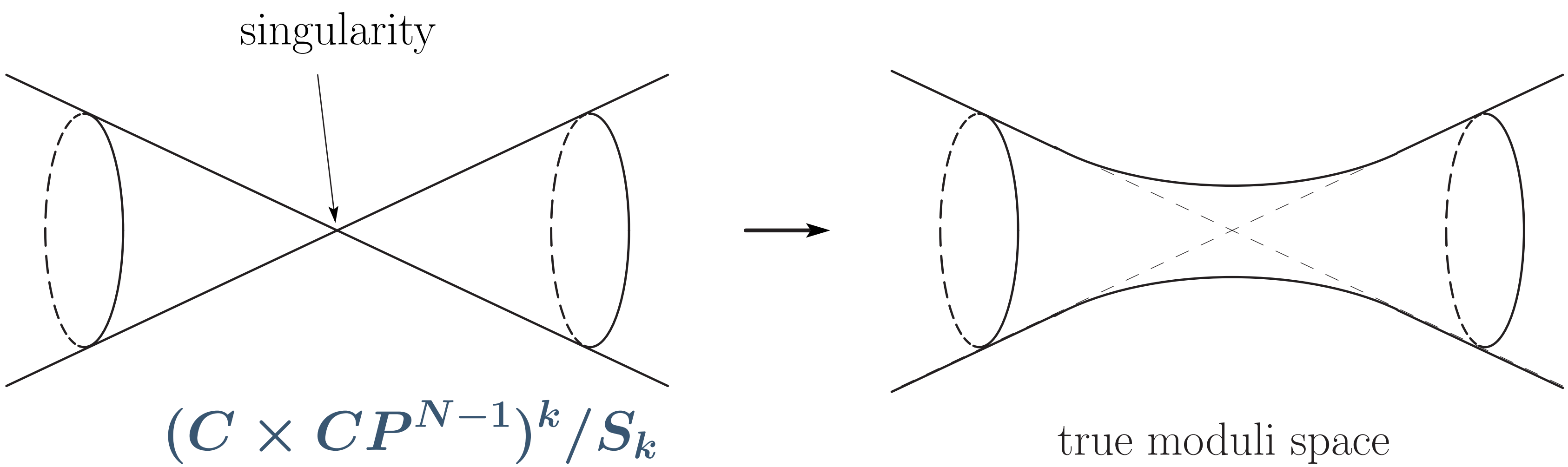


Moduli space of separated k vortices is asymptotically symm-prod of $\mathcal{M}_{k=1}$:

$$\mathcal{M}_k \sim \left(\mathbb{C} \times \mathbb{C}P^{N-1}\right)^k / S_k.$$

This has, of course, singularity at origin and not the true moduli space.

Intuitive picture:



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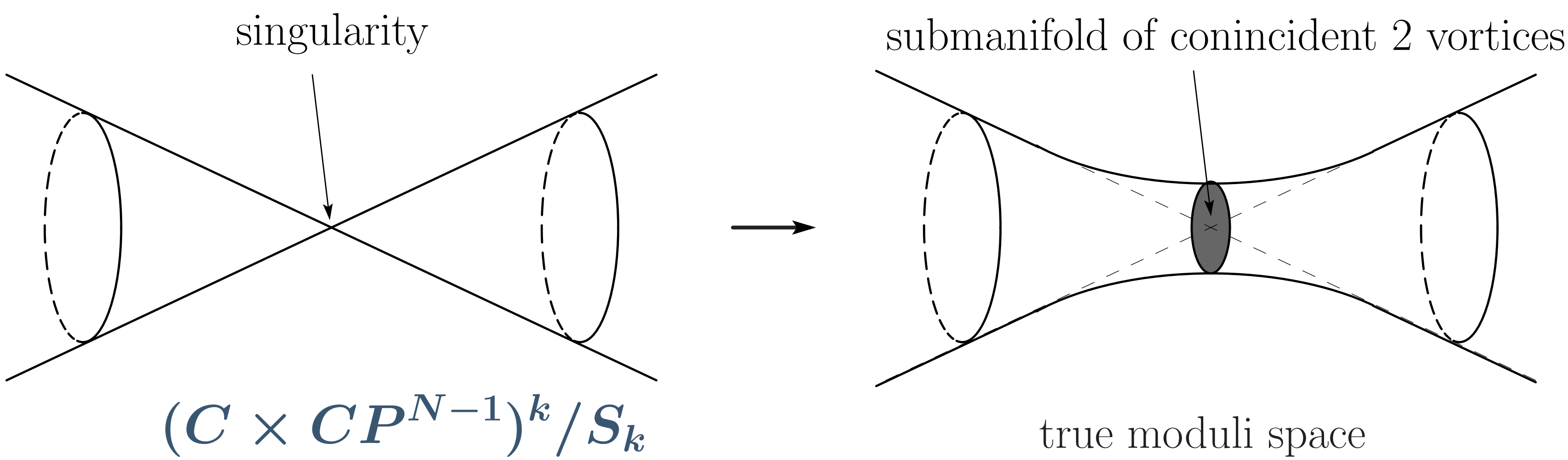

Purpose of my talk:

In my talk I will make the moduli space of coincident $k = 2$ non-Abelian vortices clear by using the moduli matrix formalism.

2. Moduli matrix formalism

Let us solve the BPS equation for the non-Abelian vortices:

$$(\mathcal{D}_1 + i\mathcal{D}_2)\, H = 0, \quad F_{12} = -\frac{g^2}{2}\left(c\mathbf{1}_N - HH^\dagger\right).$$

The first equation can be solved by

$$H(z,\bar{z}) = S^{-1}(z,\bar{z})H_0(z), \qquad A_1 + iA_2 = -iS^{-1}(z,\bar{z})\bar{\partial}_z S(z,\bar{z})$$

where we have defined the complex coordinate $z = x^1 + ix^2$.

- Moduli matrix (k vortices)
  $H_0(z)$: $N \times N$ holomophic matrix with respect to z. ($\deg[\det H_0(z)] = k$)
- $S(z,\bar{z})$ is $GL(N,\mathbb{C})$ matrix determined by second eq. for a given $H_0(z)$.

All the constant paramters contained in the moduli matrix $H_0(z)$ is moduli parateters up to an equivalence relation (called V-equivalence relation)

$$[H_0(z),\ S(z,\bar{z})] \sim V(z)\,[H_0(z),\ S(z,\bar{z})]\,, \qquad V(z) \in GL(N,\mathbb{C}).$$

$k = 1$ vortex in U(2) YM:

$$H_0^{(0,1)}(z) = \begin{pmatrix} 1 & -b \\ 0 & z - z_1 \end{pmatrix}, \qquad H_0^{(1,0)}(z) = \begin{pmatrix} z - z_1 & 0 \\ -b' & 1 \end{pmatrix}.$$

- Position of the votex: $\det H_0(z) = 0 \quad \Rightarrow \quad z = z_1$.
- Internal orientation: b or b'.
  The orientational vector is defined as a null vector of $H_0(z = z_1)$:

$$H_0(z = z_1)\vec{\phi} = \vec{0} \quad \Rightarrow \quad \vec{\phi} = \begin{pmatrix} b \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1 \\ b' \end{pmatrix}.$$

with a equivalence relation $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \sim \begin{pmatrix} \lambda\phi_1 \\ \lambda\phi_2 \end{pmatrix}$, $(\lambda \in \mathbb{C}^*)$.

$k = 2$ separated vortex in U(2) YM:

$$H_0^{(0,2)} = \begin{pmatrix} 1 & -az - b \\ 0 & z^2 - \alpha z - \beta \end{pmatrix}, \quad H_0^{(1,1)} = \begin{pmatrix} z - \varphi & -\eta \\ -\tilde{\eta} & z - \tilde{\varphi} \end{pmatrix}, \quad H_0^{(2,0)} = \begin{pmatrix} z^2 - \alpha z - \beta & 0 \\ -a'z - b' & 1 \end{pmatrix}.$$

In the (0,2) patch, we can rewrite

$$\det H_0^{(0,2)} = z^2 - \alpha z - \beta = (z - z_1)(z - z_2),$$

$$z_1 + z_2 = \alpha, \qquad z_1 z_2 = -\beta.$$

The positions of two vortices are then z_1 and z_2 ($z_1 \neq z_2$) where $\det H_0 = 0$.

Orientational vectors are defined at $z = z_1, z_2$, respectively:

$$H_0(z = z_i)\vec{\phi}_i = \vec{0} \quad \Rightarrow \quad \vec{\phi}_i = \begin{pmatrix} az_i + b \\ 1 \end{pmatrix} \simeq \mathbb{C}P^1, \qquad (i = 1, 2).$$

Notice that z_1 and z_2 are not distinctive, so we get

$$\mathcal{M}_{k=2 \text{ separated}} \simeq \{(z_1, \phi_1), (z_2, \phi_2)\} \simeq \left(\mathbb{C} \times \mathbb{C}P^1\right)^2 / S_2.$$

3. Moduli space of the coincident vortices

Let us move to the coincident $k = 2$ vortices at the origin by setting

$$z_1 = z_2 = 0 \quad \Rightarrow \quad \det H_0(z) = (z - z_1)(z - z_2) \rightarrow z^2.$$

The moduli matrices are given by

$$\widetilde{H}_0^{(0,2)} = \begin{pmatrix} 1 & -az - b \\ 0 & z^2 \end{pmatrix}, \quad \widetilde{H}_0^{(1,1)} = \begin{pmatrix} z - \varphi & -\eta \\ -\tilde{\eta} & z + \varphi \end{pmatrix}, \quad H_0^{(2,0)} = \begin{pmatrix} z^2 & 0 \\ -a'z - b' & 1 \end{pmatrix},$$

with $\varphi^2 + \eta\tilde{\eta} = 0$ (condition for coincidence).
This condition is automatically satisfied by introducing new coordinate by

$$\varphi = -XY, \quad \eta = X^2, \quad \tilde{\eta} = -Y^2.$$

(X, Y) correctly parametrizes (1,1) patch modulo Z_2 identification $(X, Y) \sim -(X, Y)$.

Then we find that the moduli space has three patches

- (0,2) patch by $(a, b) \simeq \mathbb{C}^2$
- (1,1) patch by $(X, Y) \simeq \mathbb{C}^2/Z_2$
  Origin $X = Y = 0$ is Z_2 fixed point (conical singularity).
- (2,0) patch by $(a', b') \simeq \mathbb{C}^2$

Global structure of the moduli space can be clarified by investigating the transition function between these three patches

	(a, b)	(a', b')	(X, Y)
$(a, b) =$	**	$(-a'/b'^2, 1/b')$	$(-1/Y^2, X/Y)$
$(a', b') =$	$(-a/b^2, 1/b)$	**	$(1/X^2, Y/X)$
$(X, Y) =$	$(\pm ib/\sqrt{a}, \pm i1/\sqrt{a})$	$(\pm 1/\sqrt{a'}, \pm b'/\sqrt{a'})$	**

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RED part is nothing but the orientational vector $\mathbb{C}P^1$ at the coincident 2 vortices:

$$H_0(z=0)\vec{\phi}=\vec{0} \quad \Rightarrow \quad \vec{\phi}=\begin{pmatrix} b \\ 1 \end{pmatrix} \sim \begin{pmatrix} X \\ Y \end{pmatrix} \sim \begin{pmatrix} 1 \\ b' \end{pmatrix} \simeq \mathbb{C}P^1$$

except for $X=0=Y$.

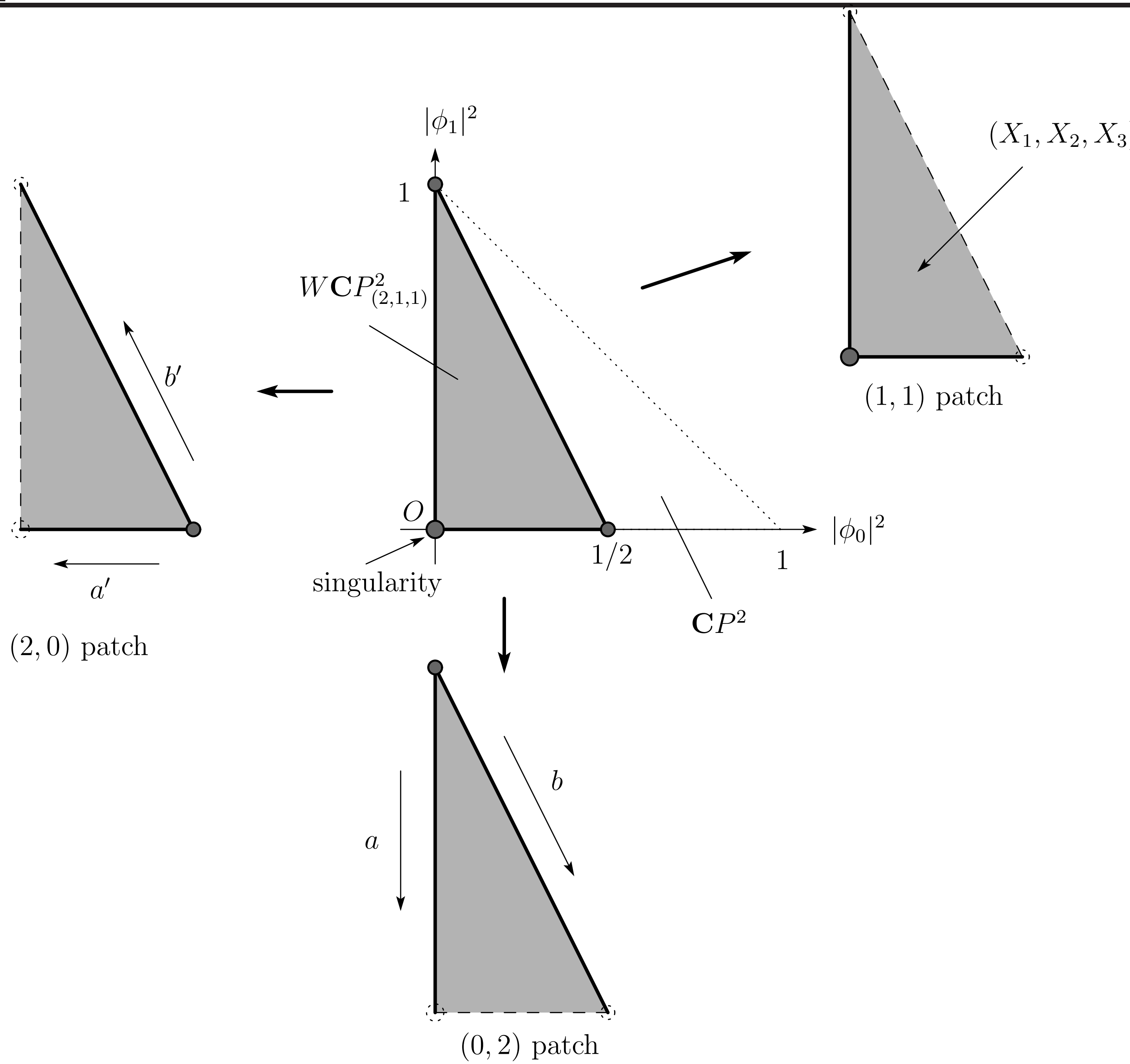
Orientational vector cannot capture all the moduli paramters, say a.
However, the whole transition functions can be written in very simple way

$$\begin{pmatrix} -a \\ b \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1 \\ X \\ Y \end{pmatrix} \sim \begin{pmatrix} a' \\ 1 \\ b' \end{pmatrix}$$

with a weighted equivalence relation "$\sim$": $\begin{pmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \end{pmatrix} \sim \begin{pmatrix} \lambda^2\phi_0 \\ \lambda\phi_1 \\ \lambda\phi_2 \end{pmatrix} \quad (\lambda \in \mathbb{C}^*)$.

Then we conclude that the submanifold of the coincident 2 vortices is

$$\widetilde{\mathcal{M}}_{k=2,N=2} \simeq W\mathbb{C}P^2_{(2,1,1)}.$$



$|\phi_1|^2$
1
$W\mathbf{C}P^2_{(2,1,1)}$
O
singularity
1/2
1
$|\phi_0|^2$
$\mathbf{C}P^2$
(X_1, X_2, X_3)
(1,1) patch
b'
a'
(2,0) patch
a
b
(0,2) patch

- We have 2 vortices but they are coincide.
  So only one orientational vector

$$\vec{\phi} = \begin{pmatrix} b \\ 1 \end{pmatrix} \sim \begin{pmatrix} X \\ Y \end{pmatrix} \sim \begin{pmatrix} 1 \\ b' \end{pmatrix} \simeq \mathbb{C}P^1, \quad (X \neq 0 \neq Y)$$

  can be defined for coincident 2 vortices unlikely separated 2 vortices.

- Physical meaning of the $\mathbb{C}P^1$ is the orientation of aligned orientational vectors.
  $\Rightarrow$ Orientation vectors of 2 coincident vortices has to be aligned.

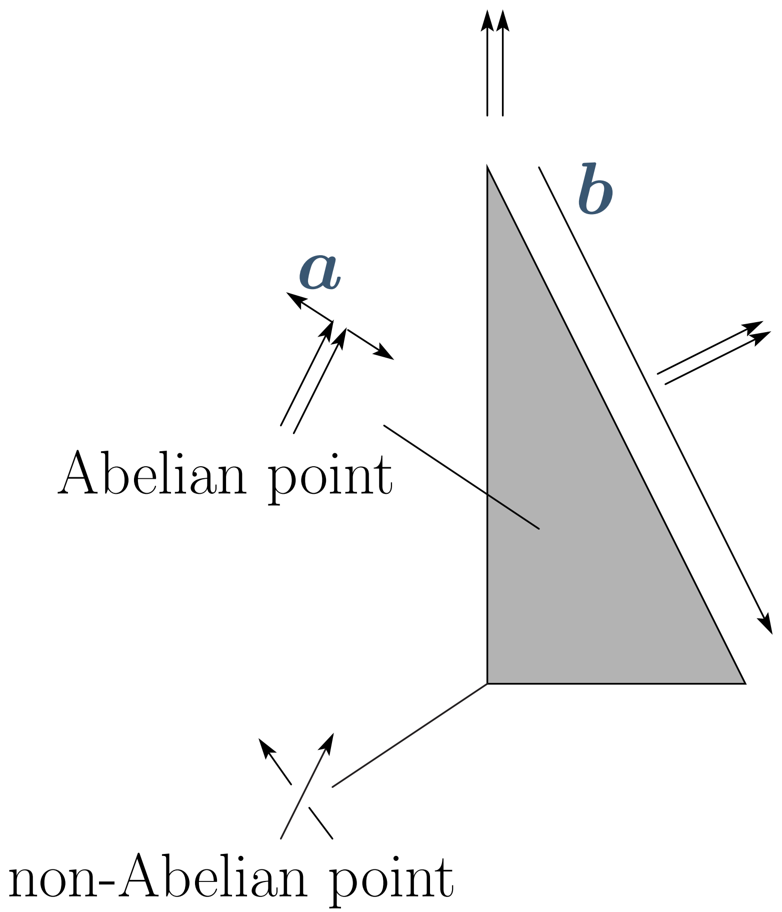
- $a(a')$ is a kind of velocity:

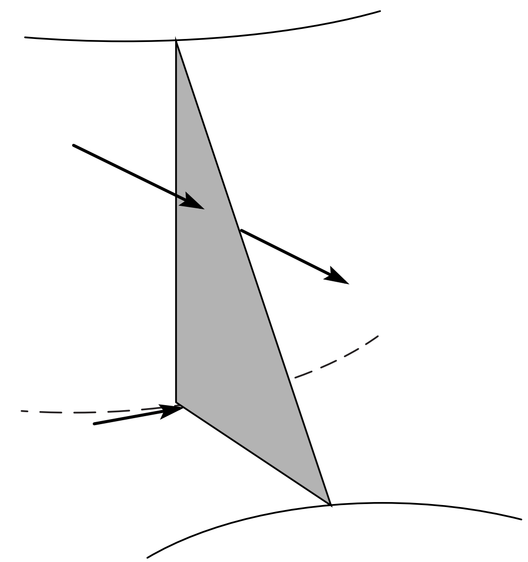
$$(a, b) \cup (a', b') \simeq T\left(\mathbb{C}P^1\right).$$

- At the singular point $X = Y = 0$, we cannot define the orientation explicitly, because the rank of the moduli matrix reduces by 2 (not 1) at that point

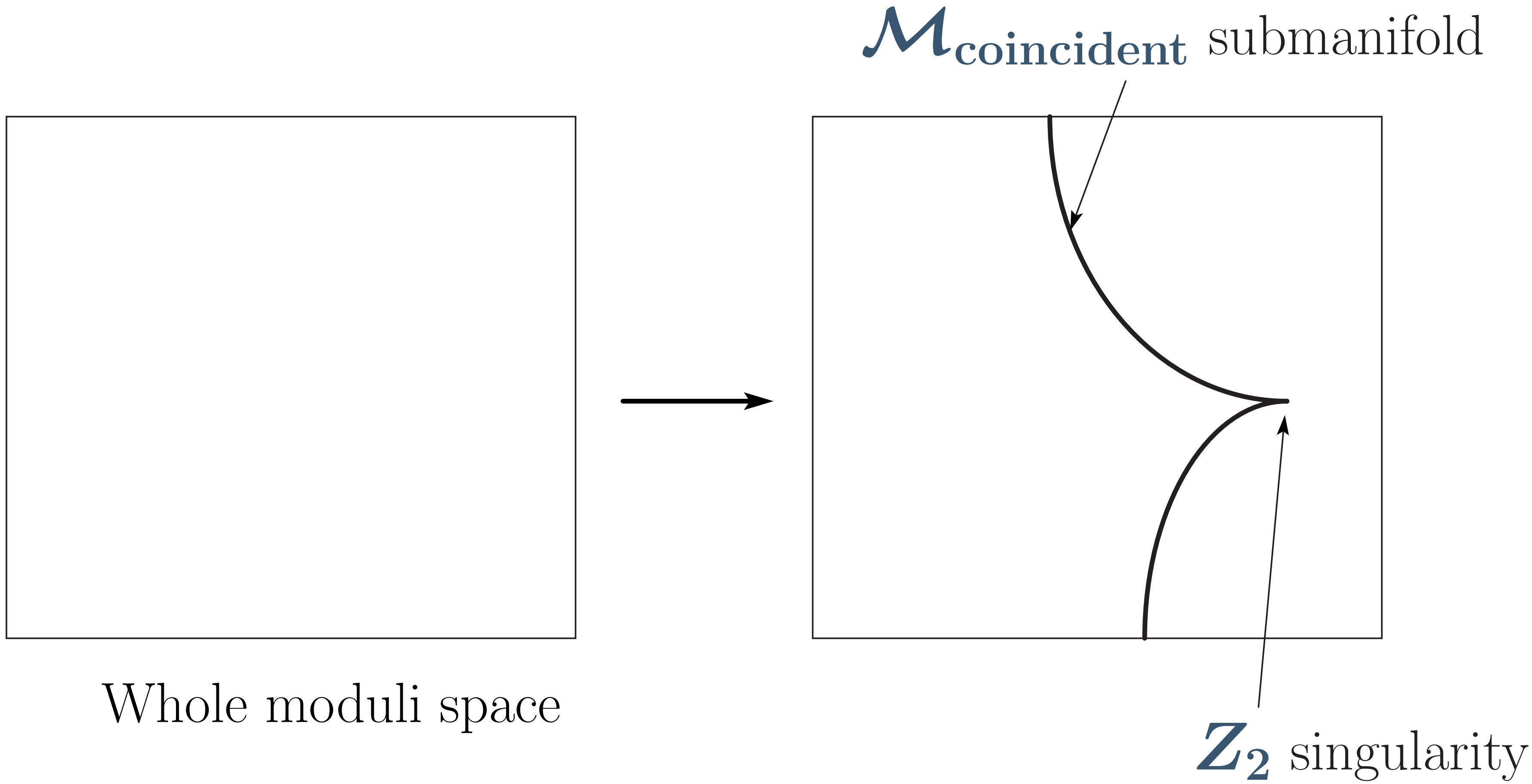
$$H_0^{(1,1)}(z=0) = \begin{pmatrix} XY & -X^2 \\ Y^2 & -XY \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\Big|_{X=0=Y}.$$

  This means that 2 coincident vortices with miss-aligned orientations is allowed only at the singularity.





head on colision of two vortices


$\mathcal{M}_{\mathbf{coincident}}$ submanifold
Whole moduli space
Z_2 singularity