Selecting Gauge Theories on an Interval
by 5D Gauge Transformations

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hep-th/0604121, Talk at Hawai meeting 2006.10.29-11.03,

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1 Introduction

Higher Dimensional **Gauge fields** in **Brane-World**

Extra-dim. component of gauge field acts as a **Higgs** scalar


Gauge symmetry breaking:

\[
\begin{cases}
\text{Wilson Lines : (VEV of extra dim comp)} \\
\text{Boundary Conditions}
\end{cases}
\]

**Orbifold** Models: [(Inner) Automorphisms of the Lie Algebra]

**Rank** of Gauge Group cannot be reduced


**Variational Principle** → wider class of boundary conditions

Consider Gauge theories on an **Interval**

Require **Vanishing surface terms** in varying the action

**Rank** of Gauge Group can be reduced
Are these Boundary Cond. Consistent with 5D Gauge Transformations?

Two Characteristics of Spontaneously broken gauge theory (in 4D)

\[ \begin{align*} 
& \text{Ward–Takahashi identity} \\
& \text{Tree Level Unitarity} 
\end{align*} \]

**Ward-Takahashi identity**

Scattering amplitudes vanish for longitudinal massless gauge boson

**Tree Level Unitarity**

\[ -i \langle n | (T - T^\dagger) | n \rangle = \sum_m \langle n | T^\dagger | m \rangle \langle m | T | n \rangle \]

→ **unitarity bound** for elastic amplitude (should not diverge)

If 2 body (in-)elastic scattering amplitudes grow with energy → violates **unitarity bound** for elastic scattering amplitude

Llewellyn Smith, Phys.Lett.**B46**, 233 (1973); Dicus-Mathur, Phys.Rev.**D7**, 3111 (1973);
5D (Higher dim.) Gauge theories

$S^1$ compactification: **KK gauge bosons** cancel growing contributions

Chivukula-Dicus-He, Phys.Lett.**B525**, 175 (2002); · · ·

Boundary Conditions from **Variational Principle**:

**Elastic scattering** amplitudes of KK modes do not grow with energy

**Results of Our Work:**

1. **Inelastic scattering** amplitudes of KK modes are computed with gauge theory on an interval

2. **Ward-Takahashi identity** and **Tree Level Unitarity**
   can be violated by **Boundary contributions**

3. Variational principle allows **boundary conditions** to give boundary contributions **violating W-T identity and tree level unitarity**

4. **Consistently defining the 5D gauge transformations**
   **forbids** such boundary conditions
2 Variational Principle and Scattering Amplitudes

Boundary Conditions from Variational Principle

Warped geometry

\[ ds^2 = g_{MN} dx^M dx^N = e^{-4W(y)} \eta_{\mu\nu} dx^\mu dx^\nu + g_{55}(y) dy^2 \]

\[ \eta_{\mu\nu} = \text{diag}(-, +, +, +), \quad 0 \leq y \leq \pi R \]

\( SU(N) \) pure gauge theory with a gauge fixing function \( G^a \)

\[ S + S_{GF} = \int d^4x \int_0^{\pi R} dy \sqrt{-g(y)} \left( -\frac{1}{4} F^a_{MN} F^a_{PQ} g^{MP} g^{NQ} - \frac{1}{2\xi} (G^a)^2 \right) \]

\[ G^a = e^{4W} \left( \eta^{\mu\nu} \partial_\mu A^a_\nu + \xi \frac{1}{\sqrt{g_{55}}} \partial_5 \frac{e^{-4W}}{\sqrt{g_{55}}} A^a_5 \right) \]

\[ F^a_{MN} = \partial_M A^a_N - \partial_N A^a_M + g_5 f^{abc} A^b_M A^c_N \]

Variation of action gives equation of motion if the surface term vanishes

\[ (\delta S + \delta S_{GF})_{\text{boundary}} = - \int d^4x \left[ \frac{e^{-4W}}{\sqrt{g_{55}}} \eta^{\mu\nu} F^a_{5\mu} \delta A^a_\nu + \frac{e^{-8W}}{\sqrt{g_{55}}} G^a \delta A^a_5 \right]_{y=\pi R} \]

\[ = \int d^4x \left[ \frac{e^{-4W}}{\sqrt{g_{55}}} \eta^{\mu\nu} F^a_{5\mu} \delta A^a_\nu + \frac{e^{-8W}}{\sqrt{g_{55}}} G^a \delta A^a_5 \right]_{y=0} \]
Possible **Boundary conditions** for **vanishing surface terms**

\[ A^a_\mu \bigg| = 0, \quad A^a_5 \bigg| = 0, \quad \text{(inconsistent)} \quad \text{[next sec.]} \]

\[ A^a_\mu \bigg| = 0, \quad \partial_5 \frac{e^{-4W}}{\sqrt{g_{55}}} A^a_5 \bigg| = 0 \quad \text{(Dirichlet)} \]

\[ \partial_5 A^a_\mu \bigg| = 0, \quad A^a_5 \bigg| = 0 \quad \text{(Neumann)} \]

Unbroken gauge symmetry is required to form a subgroup \( H \)

\[ \partial_5 A^a_\mu \bigg| = 0, \quad a \in H \subseteq G, \quad y = 0, \pi R \]

Kaluza-Klein **mode decomposition** for boundary cond \( D(a) \)

\[ A^a_\mu (x, y) = \sum_{n=0}^{\infty} A^a_{\mu n} (x) f_n^{D(a)} (y), \quad A^a_5 (x, y) = \sum_{n=0}^{\infty} A^a_{5 n} (x) g_n^{D(a)} (y) \]

**Scattering Amplitudes**

Gauge boson scattering \( A^a_n A^b_m \rightarrow A^c_i A^d_m \)

**Large** \( E \) with **fixed angle** \( \theta \), longitudinal polarization \( \epsilon \) except for \( A^c_i \)

Terms (of invariant amplitude) **growing in** \( E \) (assuming \( \xi \ll E^2/m_k^2 \))

Massive mode for the external \( A^c_i \) boson with polarization \( \epsilon^*(p_3) \)
Table 1: Kinematics: The energy of each mode is given as $E_n = \sqrt{p^2 + m_n^2}$. Both the longitudinal and transverse polarizations $\epsilon(p_3)$ are explored for the gauge field $A_i^c$.

<table>
<thead>
<tr>
<th>$p_1$</th>
<th>$\epsilon(p_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(E_n, 0, 0, p)$</td>
<td>$(p/m_n, 0, 0, E_n/m_n)$</td>
</tr>
<tr>
<td>$p_2$</td>
<td>$\epsilon(p_2)$</td>
</tr>
<tr>
<td>$(E_m, 0, 0, -p)$</td>
<td>$(p/m_m, 0, 0, -E_m/m_m)$</td>
</tr>
<tr>
<td>$p_3$</td>
<td>$\epsilon(p_3)$</td>
</tr>
<tr>
<td>$(E_l, p' \sin \theta, 0, p' \cos \theta)$</td>
<td>$(p'/m_m, -E'_m \sin \theta/m_m, 0, -E'_m \cos \theta/m_m)$</td>
</tr>
<tr>
<td>$p_4$</td>
<td>$\epsilon(p_4)$</td>
</tr>
<tr>
<td>$(E'_m, -p' \sin \theta, 0, -p' \cos \theta)$</td>
<td></td>
</tr>
</tbody>
</table>

Transverse pol $\epsilon(p_3) = (0, \cos \theta, 0, -\sin \theta)$: no growing contribution

Transverse pol $\epsilon(p_3) = (0, 0, 1, 0)$ gives an amplitude growing in $E$

$$-i g_5^2 f^{abe} f^{cde} \frac{E \sin \theta}{4 m_n m_m^2} \mathcal{K}$$
\[ K = -3 \sum_k \left[ \frac{e^{-4W}}{\sqrt{g_{55}}} g_k^{D(e)} f_m^{D(b)} f_l^{D(c)} \right]_0^{\pi R} \left[ \frac{e^{-4W}}{\sqrt{g_{55}}} g_k^{D(e)} f_m^{D(d)} f_n^{D(a)} \right]_0^{\pi R} + 2 \left[ \frac{e^{-4W}}{\sqrt{g_{55}}} \left( f_n^{D(a)} f_m^{D(b)} f_m^{D(d)} f_l^{D(c)} \right)' \right]_0^{\pi R} \]

which is a boundary contribution

Longitudinal polarization gives an amplitude growing in \( E \)

\[ -i g_5^2 f^{ab e} f^{c d e} \frac{E^2 (1 - \cos \theta)}{8 m_n m_m^2 m_l} K \]

Ward-Takahashi identity: Zero mode \((l = 0)\) for the gauge boson \( A_l^c \)

\( \epsilon^* (p_3) \to p' (1, \sin \theta, 0, \cos \theta) \) should give vanishing amplitude, but

\[ i g_5^2 f^{ab e} f^{c d e} (1 - \cos \theta) \frac{E^2}{8 m_n m_m^2 m_l} f_l^{D(e)} \left[ \frac{e^{-4W}}{\sqrt{g_{55}}} \left( f_m^{D(b)} f_m^{D(d)} f_n^{D(a)}' - f_n^{D(a)} f_m^{D(b)} f_m^{D(d)}' \right) \right]_0^{\pi R} \]

which is also a boundary contribution
Coset-N/Subgroup-D Boundary Condition

Warped geometry: $ds^2 = e^{-4W(y)} \eta_{\mu\nu} dx^\mu dx^\nu + g_{55}(y) dy^2$

$m_{\text{GUT}}$ scale at $y = 0$, $m_W$ scale at $y = \pi R$ such as ($m_{\text{GUT}} \gg m_W$)

$SU(5) \rightarrow SU(3) \times SU(2) \times U(1) \rightarrow SU(3) \times U(1)$

Boundary condition at $y = 0$ ($m_{\text{GUT}}$ scale)

$$
\begin{pmatrix}
N \\
D \\
N
\end{pmatrix}^{N_1}_{N} \quad \text{at} \quad y = 0,
$$

$SU(N) \rightarrow SU(N_1) \times SU(N - N_1) \times U(1)$

Boundary condition at $y = \pi R$ ($m_W$ scale)

$$
\begin{pmatrix}
N \\
N \\
D
\end{pmatrix}^{N_1}_{N} \quad \text{at} \quad y = \pi R,
$$

$SU(N_1) \times SU(N - N_1) \times U(1) \rightarrow SU(N_1) \times U(1)$

Variational principle allows Dirichlet condition for $SU(N - N_1)$

$
\rightarrow \text{rank reduction}
$

Neumann condition for coset is allowed (already broken at $m_{\text{GUT}}$)

We call this Coset-N/Subgroup-D boundary condition
Coset-N/Subgroup-D boundary condition gives boundary contributions


![Feynman diagrams](image)

Figure 2: Typical Feynman diagrams giving the boundary contributions violating the Ward-Takahashi identity and the tree level unitarity for \(W + X \rightarrow B(G) + X\).

Boundary conditions for \(a, b, c, d = W, X, B(G), X\)

\[
\left[ \frac{e^{-4W}}{\sqrt{g_{55}}} \left( f_n^{D(a)} f_m^{D(b)} f_m^{D(d)} f_l^{D(c)} \right)' \right]_{0}^{\pi R} \neq 0
\]

Boundary conditions for \(a, b, c, d = X, X, X, X, e = W\)

\[
\sum_{k} \left[ \frac{e^{-4W}}{\sqrt{g_{55}}} g_k^{D(e)} f_m^{D(b)} f_l^{D(c)} \right]_{0}^{\pi R} \neq 0
\]

\[
\left[ \frac{e^{-4W}}{\sqrt{g_{55}}} g_k^{D(e)} f_m^{D(d)} f_n^{D(a)} \right]_{0}^{\pi R} \neq 0
\]
Violation of W-T identity: Boundary conditions for \( a, b, c, d = W, X, B(G), X \)

\[
\left[ \frac{e^{-4W}}{\sqrt{g_{55}}} \left( f_m D^{(b)} f_m D^{(d)} f_n D^{(a)} - f_n D^{(a)} f_m D^{(b)} f_m D^{(d)} \right) \right]^{\pi R}_0 \neq 0
\]

Figure 3: Another typical Feynman diagrams giving the boundary contributions violating the Ward-Takahashi identity and the tree level unitarity for \( X + X \rightarrow X + X \).

3 Consistency of 5D Gauge Transformations

5D Gauge Transformations

5D gauge transformations for 4D vector component \( A^a_\mu(x, y) \)

\[
\delta A^a_\mu(x, y) = \partial_\mu \epsilon^a(x, y) + g_5 f^{abc} A^b_\mu(x, y) \epsilon^c(x, y)
\]
inhomogeneous term = indep of \( g_5 \),

nonlinear term = first order in \( g_5 \)

5D gauge transformations for 4D scalar component \( A_5^a(x, y) \)
\[
\delta A_5^a(x, y) = \partial_5 \epsilon^a(x, y) + g_5 f^{abc} A_5^b(x, y) \epsilon^c(x, y)
\]

Opposite Boundary Cond for \( A_\mu, A_5 \)

inhomogeneous term →
\[
A_\mu^a = \epsilon^a = 0 \text{ or } \partial_5 A_\mu^a = \partial_5 \epsilon^a = 0 \text{ (same boundary condition)}
\]
\[
A_5^a = \partial_5 \epsilon^a = 0 \text{ or } \partial_5 \frac{e^{-4W}}{\sqrt{g}} \epsilon^a = \epsilon^a = 0 \text{ (opposite boundary condition)}
\]
→ opposite boundary conditions for \( A_\mu^a \) and \( A_5^a \)

Coset-N/Subgroup-D Boundary Condition

Nonlinear term \( g_5 f^{abc} A_\mu^b \epsilon^c \in \delta A_\mu^a \)

1. If \( A_\mu^b \epsilon^c \neq 0 \), then \( \delta A_\mu^a \neq 0 \):
   
   Dirichlet condition is not satisfied by \( A_\mu^a \)
2. If $\partial_5 A^b_\mu \epsilon^c \neq 0$ or $A^b_\mu \partial_5 \epsilon^c \neq 0$, then $\partial_5 \delta A^a_\mu \neq 0$:

Neumann condition is not satisfied by $A^a_\mu$

Case 1. occurs if $(a, b, c) = (W_\mu, X_\mu, X_\mu)$: $(D, N, N)$ boundary cond.

$X_\mu \epsilon^X \neq 0, \rightarrow \delta W_\mu \neq 0$,

Case 2. occurs

if $(a, b, c) = (X_\mu, X_\mu, W_\mu)$: $(N, N, D)$ boundary cond.

$\partial_5 W_\mu \epsilon^X \neq 0, \rightarrow \partial_5 \delta X_\mu \neq 0$,

if $(a, b, c) = (X_\mu, W_\mu, X_\mu)$: $(N, D, N)$ boundary cond.

$X_\mu \partial_5 \epsilon^W \neq 0, \rightarrow \partial_5 \delta X_\mu \neq 0$,

5D gauge transformations cannot be defined consistently with Coset-N/Subgroup-D Boundary Condition

Similarly for $A_5$

4 Summary and Future Problems

1. Variational principle allows boundary conditions that violate the Ward-Takahashi identity and the tree level unitarity.
2. **5D gauge transformation** parameters $\epsilon(x, y)$ besides the gauge fields $A_M(x, y)$ should be given a **boundary condition** which is **consistent with the 5D gauge transformations**.

3. This condition provides a stringent constraint and forbids these boundary conditions that violate the tree level unitarity and the Ward-Takahashi identity.

**Future problems**

1. Consistency of 5D gauge transformations $\rightarrow$
   W-T identity and tree-level unitarity are not yet checked for all processes
2. More general boundary conditions than automorphisms in orbifolding ?
3. Nondiagonal automorphisms of the Lie algebra: interesting possibility
   - Hebecker and March-Russell, Nucl.Phys.**B625**, 128 (2002); · · ·
4. Deconstruction: consistent 5D gauge transformations usually assured.
   Can all consistent boundary conditions be realized by deconstruction ?