Quantum boundary conditions and particle bound states

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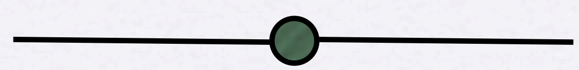
Preface

- Exotic, "field theoretical" phenomena in solvable quantum mechanics -- Anomaly, duality, anholonomy
- Generalized δ-potentials in one dimension
- Topological structure $U(N)$ of parameter space
- Extensions to two and three dimensions
- Applications to nuclear spectra

Structure of Parameter Space

- Most general point interaction
- $\mathrm{H} = -\frac{1}{2}\frac{d^2}{dx^2}$ self-adjoint on $\mathbb{R} \setminus \{0\}$



x=0

- boundary condition

$$\Phi \equiv \begin{pmatrix} \varphi(0_+) \\ \varphi(0_-) \end{pmatrix} \qquad \Phi' \equiv \begin{pmatrix} \varphi'(0_+) \\ -\varphi'(0_-) \end{pmatrix}$$

- $\Phi^+\Phi' = \Phi'^+\Phi \quad -> \quad |\Phi + iL_0\Phi'| = |\Phi + iL_0\Phi'|$

$S_1 x S_3$ Manifold

- $(U-I)\Phi + iL_0(U+I)\,\Phi' = 0$
- $U \equiv e^{i\xi}\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$; $\alpha_r^2+\alpha_i^2+\beta_r^2+\beta_i^2=1$
- Ω ; U(2)=U(1)xSU(2) 3-sphere
- If $\beta_r\beta_i \neq 0$, local representation

$$\begin{pmatrix} \varphi(0_+) \\ \varphi'(0_+) \end{pmatrix} = e^{i\lambda}\begin{pmatrix} a & d \\ b & c \end{pmatrix}\begin{pmatrix} \varphi(0_-) \\ \varphi'(0_-) \end{pmatrix}$$

$ac-bd=1$

U(1)xSL(2,R)

Exotic features

- Existence of “2nd class” δ-potential that induces discontinuity in wave function: “ε-potential”

- High- and low-pass quantum filter

- Parity-duality with reversed coupling
  : bosons with δ-potential = fermions with ε-potential

- Anholonomy in eigenstate (Berry phase)

- Anholonomy in eigenvalues (spiral)

Symmetry Transformation

- $\Sigma : H_U \rightarrow H_{U'} = \Sigma H_U \Sigma^{-1}$: isospectral

$$H_{U'}[\Sigma\varphi(x)] = \Sigma H_U \varphi(x) = E\,[\Sigma\varphi(x)]$$

$\varphi(x) \rightarrow \Sigma\varphi(x)$
$U \rightarrow U' = \sigma U \sigma$: σ 2x2 unitary
$\sigma = [c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_3]e^{i\lambda}$

- σ-invriant submanifolds (torus)
  eg. $U = \sigma_3 U \sigma_3$ then $U = a_0 + a_3\sigma_3 = \begin{pmatrix} e^{i\theta_+} & 0 \\ 0 & e^{i\theta_-} \end{pmatrix}$

Spectral Decomposition

- Spectra of H_U : eigenvalue of U
- $U = V^{-1} D V$: $D = e^{i\xi} e^{i\rho\sigma_3}$
  $V = e^{i\mu/2\,\sigma_2} e^{i\nu/2\,\sigma_3}$
- Spectra of H_U : eigenvalue of U
  : σ exists such that $U = \sigma D \sigma$

$$D = \begin{pmatrix} e^{i\theta_+} & 0 \\ 0 & e^{i\theta_-} \end{pmatrix}$$

$$\theta_\pm = \xi \pm \rho \in [0, 2\pi)$$

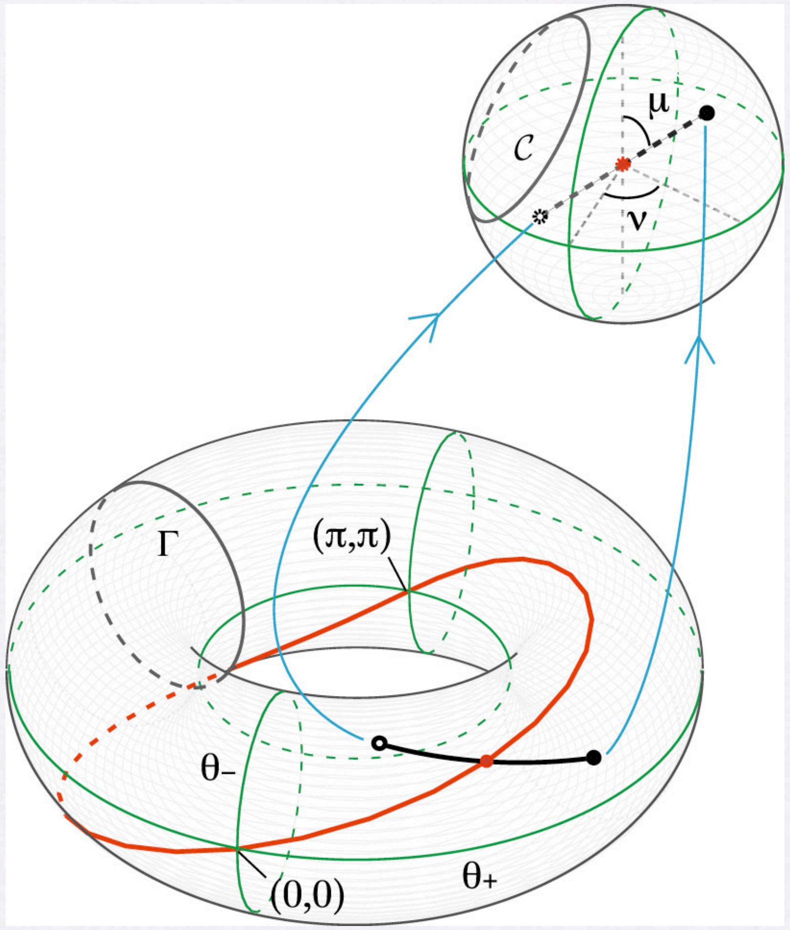
$$\mu \in [0, \pi] \,, \nu \in [0, 2\pi) \quad \text{:Polar}$$

Spectral Decomposition

- $\Omega = \{ (\theta_+,\theta_-,\mu,\nu) \}$
  :angular paramtrization

- $\Omega_{sp} = \{ (\theta_+,\theta_-) \}$; T^2 Torus

- $\Omega_{iso} = \{ (\mu,\nu) \}$; S^2 Sphere



Separated Torus

- For $\beta_r = \beta_i = 0$

- $U = e^{i\xi}\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$ torus U(1)xU(1)⊂U(2)

- With $\chi_\pm = \xi \pm \phi$, connection separates

$$\sin(\chi_+/2)\varphi(0_+)+\cos(\chi_+/2)\varphi'(0_+)=0$$
$$\sin(\chi_-/2)\varphi(0_-)+\cos(\chi_-/2)\varphi'(0_-)=0$$

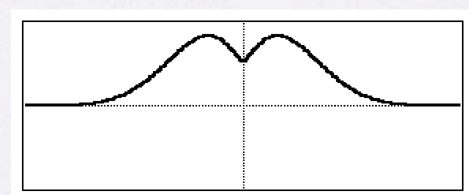
- This cond. obtained as $\sigma_3 U \sigma_3 = U$

Semi-Reflection Symmetry

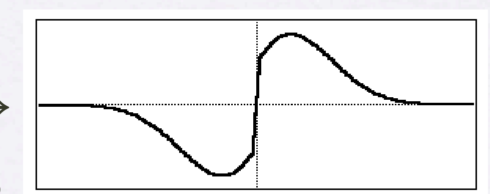
- $\mathcal{P}_3$: $\varphi(x) \rightarrow \mathcal{P}_3\, \varphi(x) := (\Theta(x) - \Theta(-x))\, \varphi(x)$

$$\Phi \xrightarrow{\mathcal{P}_3} \sigma_3 \Phi$$

$$\Phi' \xrightarrow{\mathcal{P}_3} \sigma_3 \Phi'$$



$\xrightarrow{\;}$ $\mathcal{P}_3$

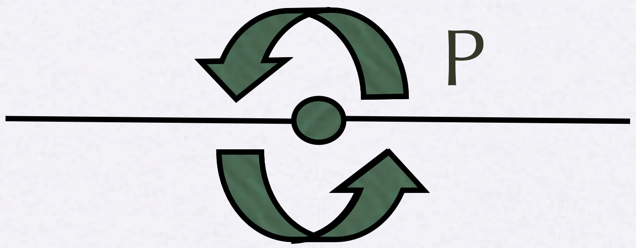


- P3-invariant torus
  $\Omega_{\mathcal{P}3}$; $T^2 = S^1 x S^1 \subset \Omega$: $\sigma_3 U \sigma_3 = U$

Parity Invariant T^2

- $\mathcal{P}_1$: $\varphi(x) \to \mathcal{P}_1\, \varphi(x) := \varphi(-x)$

$$\Phi \xrightarrow{\mathcal{P}_1} \sigma_1 \Phi$$
$$\Phi' \xrightarrow{\mathcal{P}_1} \sigma_1 \Phi'$$



- $\mathcal{P}_1$-invariant torus
Ω_{P1} ;$T^2 = S^1 x S^1 \subset \Omega$: $\sigma_1 U \sigma_1 = U$

- $U = e^{i\xi} \begin{pmatrix} \cos\phi & i\sin\phi \\ i\sin\phi & \cos\phi \end{pmatrix}$; $\alpha_i = \beta_r = 0$

Algebra su(2)

- $\mathcal{P}_2$: $\varphi(x) \to \mathcal{P}_2\, \varphi(x) := i(\Theta(-x)-\Theta(x))\varphi(-x)$

$$\Phi \xrightarrow{\mathcal{P}_2} \sigma_2\Phi$$
$$\Phi' \xrightarrow{\mathcal{P}_2} \sigma_2\Phi'$$

- su(2) algebra on $L(R)\otimes C^2$
  - $\mathcal{P}_1{}^2=\mathcal{P}_2{}^2=\mathcal{P}_3{}^2=1$, $\mathcal{P}_1\mathcal{P}_2=-\mathcal{P}_2\mathcal{P}_1=i\mathcal{P}_3$ etc.
- $\mathcal{P}_i\mathcal{P}_j$-invariant ring C^1 (2-fold degeneracy)
  Ω_{SD} ; $S^1 \subset \Omega_{\mathcal{P}_i}$: $\sigma_j U \sigma_j = \sigma_i U \sigma_i = U$

Weyl-scale Invariant S^2

- W_λ: $\varphi(x) \to W_\lambda\varphi(x) := \lambda^{1/2}\varphi(\lambda x)$

  $$\varphi'(x) \xrightarrow{\mathcal{W}_\lambda} \lambda^{3/2}\varphi'(\lambda x)$$

  satisfy $(U-I)\Phi + iL_0(U+I)\Phi' = 0$ iff

- $\det(U+I) = \det(U-I) = 0$

- Ω_W ; $S^2 \subset \Omega$: Ω_λ-invariant 2-sphere

- $U = i\begin{pmatrix} i\alpha_i & \beta_r + i\beta_i \\ -\beta_r + i\beta_i & -i\alpha_i \end{pmatrix}$; $\alpha_i = 0$, $\xi = \pi/2$

Other Sub-manifolds

- $\mathcal{T}$: $\varphi(x) \rightarrow \mathcal{T}\varphi(x) := \varphi^*(x)$
  $\Omega_{\mathcal{T}}$; $S^1 x S^2$; $U(2)/U(1) \subset \Omega$

- $\mathcal{PT}$: $\varphi(x) \rightarrow \mathcal{PT}\varphi(x) := \varphi^*(-x)$
  $\Omega_{\mathcal{PT}}$; $S^1 x S^2$; $U(2)/U(1) \subset \Omega$

- Complimentary Manifolds
  Ω_W ; S^2 ; $U(2)/[U(1)xU(1)]$
  $U = U_W U_{\mathcal{P}3}$

Structure of Parity T^2

- $U = e^{i\xi}\begin{pmatrix} \cos\phi & i\sin\phi \\ i\sin\phi & \cos\phi \end{pmatrix}$

  $\varphi_{\pm}(0-)=\pm\varphi_{\pm}(0_{+})$

  $\varphi'_{\pm}(0-)=\mu\varphi'_{\pm}(0_{+})$

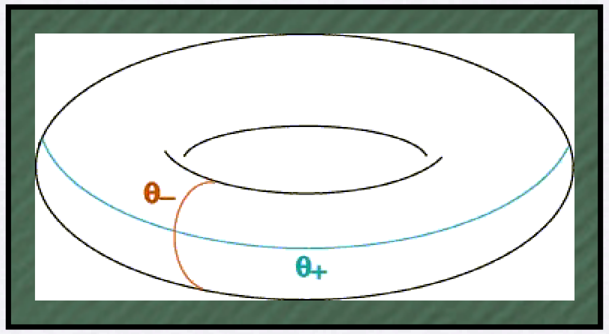
- Even/odd separate with $\theta_{\pm}=\xi\pm\phi$,

  $\sin(\theta_{+}/2)\varphi_{+}(0_{+})+\cos(\theta_{+}/2)\varphi'_{+}(0_{+})=0$

  $\sin(\theta_{-}/2)\varphi_{-}(0_{+}) +\cos(\theta_{-}/2)\varphi'_{-}(0_{+}) =0$

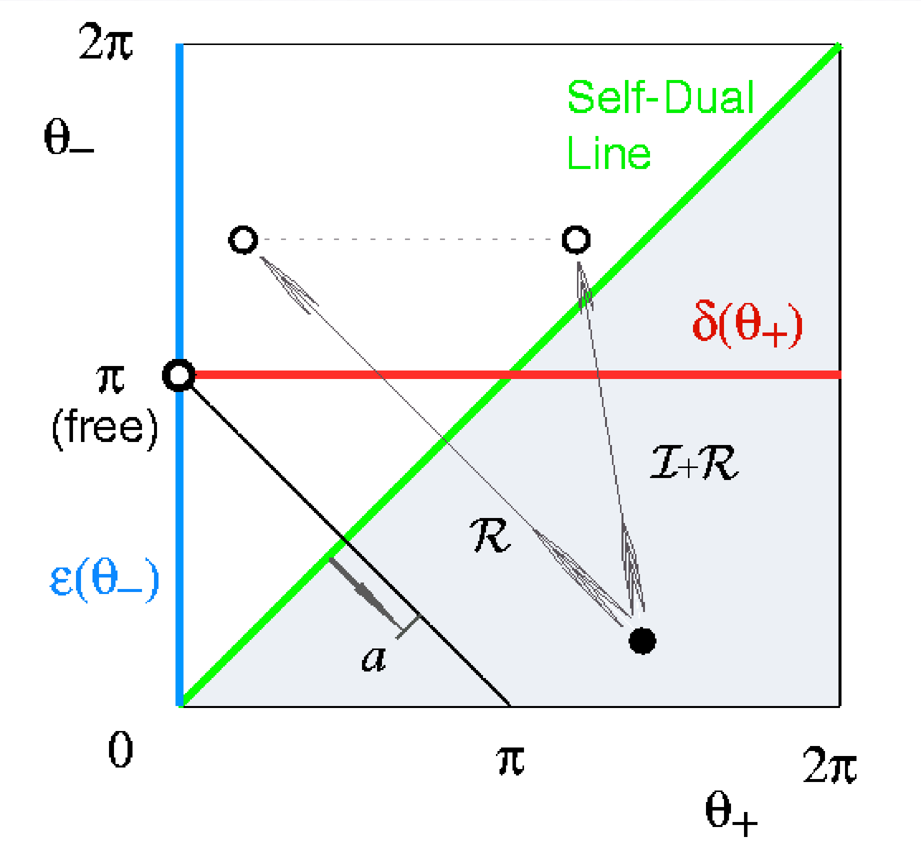
- $\theta_{-} =0, \quad \delta : \gamma_{+}=\tan(\theta_{+}/2)$

  $\theta_{+}=0, \quad \varepsilon : \gamma_{-}=\cot(\theta_{-}/2)$



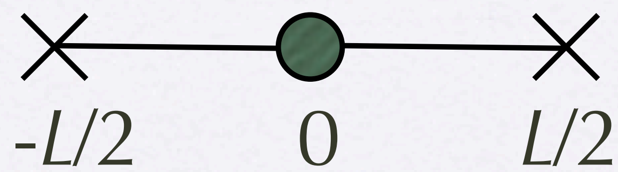
Duality on Parity Torus

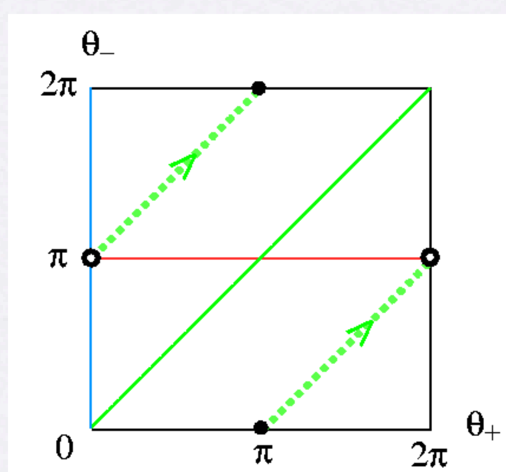
- $\mathcal{P}_3$: θ- ⟵⟶ θ+
  isospectral
- Ω_{SD} : diagonal line
- I_- : $\theta_- \to \theta_- + \pi$
  I_+: $\theta_+ \to \theta_+ + \pi$
- $\mathcal{P}_3\, I_- = I_+\, \mathcal{P}_3$:
  $\delta(\gamma) \to \varepsilon(1/\gamma)$

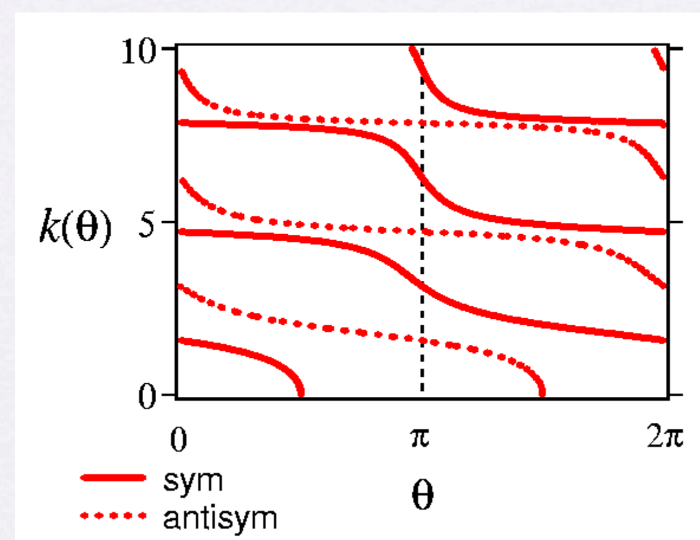


Spiral Anholonomy

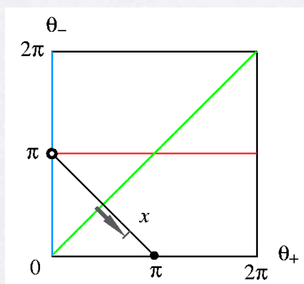
- Line of length *L* : Dirichlet edges
- U in $\Omega_{\mathcal{P}1}$; $S^1 x S^1$
- $k_+\cot k_+L=\tan(\theta_+/2)$
  $k_-\cot k_-L=\tan(\theta_-/2)$

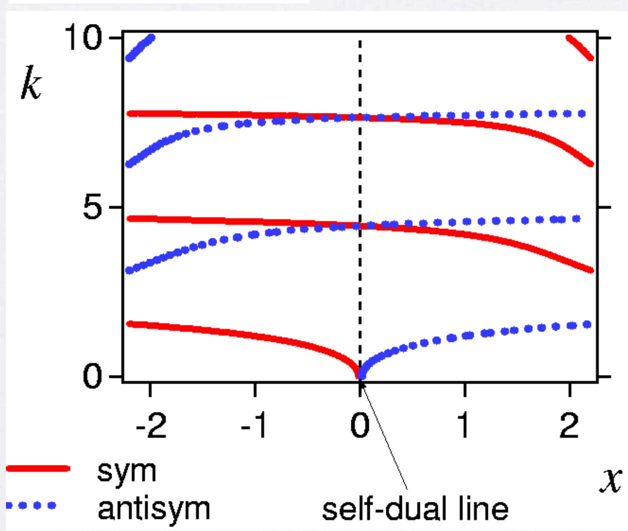




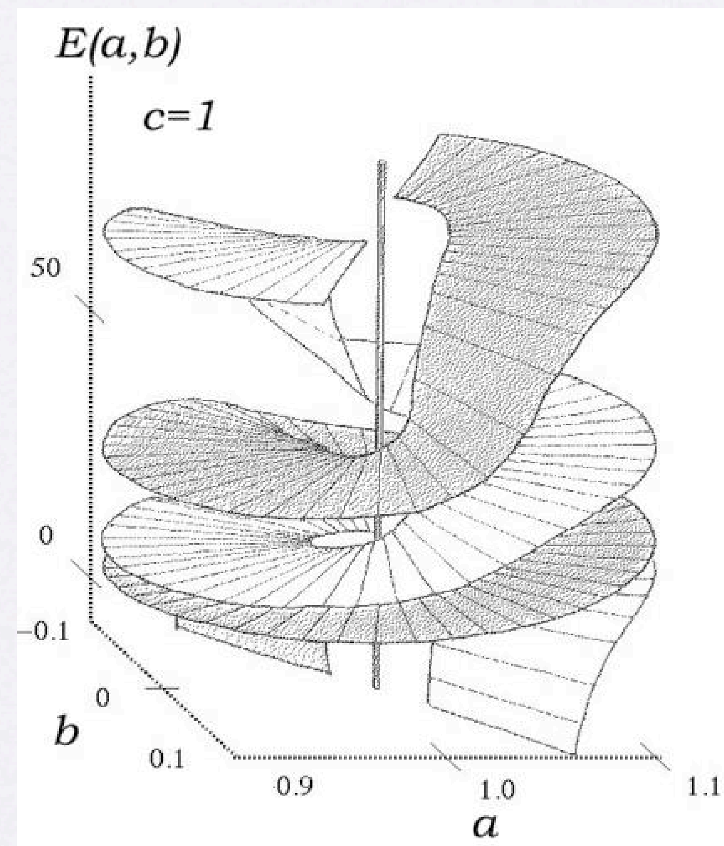


Spiral Anholonomy





Duality & Spiral



Dirac Monopole

- U in Ω_W ; S^2 ; $\pm L$: Dirichlet edges

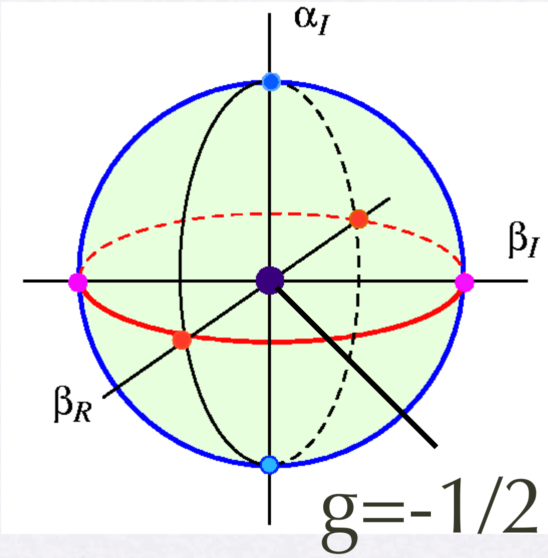
- $k\cot kL=[\sin\xi+\sqrt{(1-\alpha_r^2)}]/[\cos\xi+\alpha_r]=0$

$$\alpha_i = \cos\theta,\ \ \beta_r = \cos\theta\cos\phi,\ \ \beta_i = \cos\theta\sin\phi$$

- Berry Phase $\gamma(C) = \int_C dA$

- Connection

$$dA = \varphi i\partial_\theta\varphi\ d\theta + \varphi i\partial_\phi\phi\ d\phi$$
$$= -1/2\cdot\sin\theta d\theta d\phi$$



Extensions

- Point interaction in the presence of singular potential $\sim 1/x^p$ $(p<2)$ formulated essentially in same manner
- Point-like interaction in the presence of singular potential $\sim 1/x^p$ $(p>=2)$ can be formulated, if "point" is replaced by arbitrarily small non-zero range Δ
- Point interaction in 2 and 3 dimension can be formulated with partial wave dicomposition

Summary

- In 1D quantum mechanics, the existence of 2nd class contact force introduces nontrivial topology.
- That in turn results in various exotic phenomena in solvable quantum mechanics that are usually found in field theory.
- With slight extension, nontrivial point-like interaction can be formulated in 2D and 3D quantum mechanics.

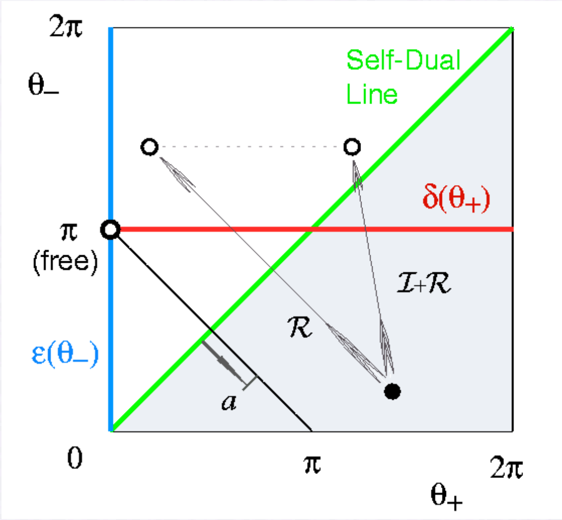
References

- T.Cheon HomePage

  http://www.mech.kochi-tech.ac.jp/cheon

Duality Transformation

- $D = \begin{pmatrix} e^{i\theta_+} & 0 \\ 0 & e^{i\theta_-} \end{pmatrix}, \quad D' = \begin{pmatrix} e^{i\theta_-} & 0 \\ 0 & e^{i\theta_+} \end{pmatrix}$ same spectra
- Spectral torus Ω_{sp} : double covering
- $\Omega(D) = \{V^{-1}DV | V \in SU(2)\}$ and
  $\Omega(D') = \{V^{-1}D'V | V \in SU(2)\}$
  are the same



- S^1 ; $\Omega_{SD} = \{\theta \mid \theta = \theta_+ = \theta_-\}$
  doubly degenerate

Moebius Strip Spectral Space

- Fundamental domain of spectral space is

  Moebius strip with boundary

- $\Sigma = T^2/Z_2$