T-duality of ZZ-branes

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Introduction

T-duality: known to hold at each order in perturbation theory
quite characteristic of string theory

$\Downarrow$

Basic question:

- **T-duality is relevent even for nonperturbative formulation of string theory?**
- **how T-duality is formulated nonperturbative way?**

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Aim: clarify the T-duality of nonperturbative effect in nonperturbative framework

This kind of study should be important for understanding the role of T-duality even in nonperturbative formulation of critical string theory.

Let us consider $N \times N$ two-matrix model (A: $\uparrow$, B: $\downarrow$)

$$S = N\mathrm{tr}V(A,B), \quad V(A,B) = V(A) + V(B) - cAB, \quad V(x) = \frac{1}{2}x^2 - \frac{g}{3}x^3,$$

Then the partition function can be written by eigenvalues of A and B:

Itzykson-Zuber, Mehta

$$\begin{aligned} Z &= \int dAdBe^{-S} \\ &= \int d\lambda_1 \cdots d\lambda_N d\mu_1 \cdots d\mu_N \Delta^{(N)}(\lambda_1 \cdots \lambda_N)\Delta^{(N)}(\mu_1 \cdots \mu_N) \\ &\quad \times \exp\left(-N\sum_{i=1}^{N} V(\lambda_i) - N\sum_{i=1}^{N} V(\mu_i) + N\sum_{i=1}^{N} c\lambda_i\mu_i\right), \end{aligned}$$

Now let us consider situation one of eigenvalues (λ_N, μ_N) is separated from others and it is outside the cut (support of the eigenvalue distribution). We can define the effective action for λ_N, μ_N as dependence of the partition function on them:

$$
\begin{aligned}
Z_N &= \int dx \int dy \int d\lambda_1 \cdots d\lambda_{N-1} \int d\mu_1 \cdots d\mu_{N-1} \\
&\times \left(\Pi_{i=1}^{N-1}(x-\lambda_i)\right)\left(\Pi_{i=1}^{N-1}(y-\mu_i)\right)\Delta^{(N-1)}(\lambda)\Delta^{(N-1)}(\mu) \\
&\times \exp\left[-N\left(\sum_{i=1}^{N-1}V(\lambda_i)+\sum_{i=1}^{N-1}V(\mu_i)-\sum_{i=1}^{N-1}c\lambda_i\mu_i+V(x)+V(y)-cxy\right)\right] \\
&\propto \int dxdy\, \langle \det(x-A')\det(y-B')\rangle_{N-1}\, e^{-N(V(x)+V(y)-cxy)} \\
&\propto \int dxdy\; e^{-NV_{\text{eff}}(x,y)}.
\end{aligned}
$$

In the large-N limit

$$
\begin{aligned}
V_{\text{eff}}^{(0)}(x,y) &= V(x)+V(y)-cxy-\left\langle\frac{1}{N}\operatorname{tr}\log(x-A)\right\rangle_d-\left\langle\frac{1}{N}\operatorname{tr}\log(y-B)\right\rangle_d \\
&= V(x)+V(y)-cxy-\int_{x_*}^{x}R_A(x')dx'-\int_{y_*}^{y}R_B(y')dy'
\end{aligned}
$$

$R_A(x)$, $R_B(y)$: resolvent for A、B (same functional form)

$$
R_A(x)=\left\langle\frac{1}{N}\operatorname{tr}\frac{1}{x-A}\right\rangle_d,\quad R_B(y)=\left\langle\frac{1}{N}\operatorname{tr}\frac{1}{y-B}\right\rangle_d
$$

$V_{\text{eff}}(x,y)$ has three saddle points: instantons in this two-matrix model

Substituting these saddle point values into $V_{\text{eff}}^{(0)}$, we get three nonperturbative effects

$$\exp\left[-\frac{8\sqrt{3}}{\sqrt{7}g_s}\right], \quad \exp\left[-\frac{4\sqrt{6}}{\sqrt{7}g_s}\right], \quad \exp\left[-\frac{4\sqrt{6}}{\sqrt{7}g_s}\right].$$

These agree with predictions from string equation

Eynard-Zinn-Justine

$$u^3 - \frac{3}{4}g_s^2 u\ddot{u} - \frac{3}{8}g_s^2\dot{u}^2 + \frac{1}{24}g_s^4 f^{(4)} = t.$$

and ZZ-brane tensions:

$$(1,1) \text{ boudary condition } : \frac{4\sqrt{6}}{\sqrt{7}g_s} \leftarrow h_{1,1} = 0$$

$$(1,2) \text{ boudary condition } : \frac{8\sqrt{3}}{\sqrt{7}g_s} \leftarrow h_{1,2} = 1/16$$

$$(1,3) \text{ boudary condition } : \frac{4\sqrt{6}}{\sqrt{7}g_s} \leftarrow h_{1,3} = 1/2$$

ZZ-branes=instantons in the two-matrix model

Kazakov-Kostov

Ishibashi-T.K.-Yamaguchi

T-duality: Kramers-Wannier duality (S-duality) on world sheet

Essentially $G_{\mu\nu} \to G^{-1}_{\mu\nu}$ (Fourier transf.):

$$
\begin{aligned}
S = & \frac{1}{4\pi}\int d^2 z G(X)\partial\theta\bar{\partial}\theta \\
\leftrightarrow \quad S' = & \frac{1}{4\pi}\int d^2 z (G^{-1}(X)V\tilde{V} + \theta(\partial\tilde{V} - \bar{\partial}V)) \\
\leftrightarrow \quad S_D = & \frac{1}{4\pi}\int d^2 z G^{-1}(X)\partial\theta_d\bar{\partial}\theta_d, \quad V = \partial\theta_d, \quad V = \bar{\partial}\theta_d,
\end{aligned}
$$

which is nothing but the Kramers-Wannier duality.
For circle compactified string, this gives $R \leftrightarrow 1/R$ (inverse temp.)

T-duality at the nonperturbative level: c=1/2 string theory

The original $c = 1/2$ string theory (Ising model on the random surface) is defined as the double scaling limit of the two-matrix model:

$$
S(A, B) = \mathrm{tr}\left(\frac{1}{2}A^2 - \frac{g}{3}A^3 + \frac{1}{2}B^2 - \frac{g}{3}B^3 - cAB\right). \quad \text{original model}
$$

A, B: up and down spin on the random surface.
Let us perform the KW transf. on the random surface.

Matrix model propagator $\leftrightarrow$ Boltzmann weight for Ising model:

$$\left.\begin{array}{lcl} \langle AA\rangle = \langle BB\rangle = Le^{\beta} & : & \text{stick} \\ \langle AB\rangle = Le^{-\beta} & : & \text{flip} \end{array}\right\} \text{of original spin}, \quad c = e^{-2\beta}, \quad L = \frac{\sqrt{c}}{1-c^2}.$$

Z_2 Fourier transf.:

$$e^{\beta} = K(e^{\tilde{\beta}} + e^{-\tilde{\beta}}), \quad e^{-\beta} = K(e^{\tilde{\beta}} - e^{-\tilde{\beta}}).$$

T-duality transf. amounts to find a matrix model with propagators $e^{\pm\tilde{\beta}}$. It is easy to see that the new matrices X, Y defined as

$$X = \frac{A+B}{\sqrt{2}}, \quad Y = \frac{A-B}{\sqrt{2}},$$

have the desired propagator:

$$\left.\begin{array}{lcl} \langle XX\rangle = \frac{1}{\sqrt{1-c^2}}e^{\tilde{\beta}} & : & \text{stick} \\ \langle YY\rangle = \frac{1}{\sqrt{1-c^2}}e^{-\tilde{\beta}} & : & \text{flip} \end{array}\right\} \text{of dual spin}$$

Thus we arrive at the dual two-matrix model

$$S_D(X,Y) = \mathrm{tr}\left(\frac{1-c}{2}X^2 + \frac{1+c}{2}Y^2 - \frac{\hat{g}}{3}(X^3 + 3XY^2)\right), \quad \hat{g} = \frac{g}{\sqrt{2}},$$

: $O(1)$ loop gas model on a random surface Kostov, Duplantier and Kostov

Note that the partition function is the same, but the fundamental correlators are different:

$$\left\langle \frac{1}{N}\mathrm{tr}\frac{1}{x-A}\right\rangle + \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{x-B}\right\rangle \not\propto \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{x-X}\right\rangle.$$

Nonperturbative effects in the dual model

$$
\begin{aligned}
Z &= \int dX dY e^{-NS_D(X,Y)} \\
&\propto \int d\lambda_i d\mu_i \frac{\Delta^{(N)}(\lambda)\Delta^{(N)}(\mu)}{\Pi_{i>j}(\mu_i+\mu_j)} \\
&\times \exp\left[-N\left(\sum_i \frac{1-c}{2}\lambda_i^2 + \sum_i \frac{1+c}{2}\mu_i^2 - \frac{\hat{g}}{3}(\sum_i \lambda_i^3 + 3\sum_i \lambda_i \mu_i^2)\right)\right],
\end{aligned}
$$

As before let us concentrate one of eigenvalues (say $x = \lambda_N$, $y = \mu_N$)) and derive the effective action for them:

$$
\begin{aligned}
Z &= N \int dxdy \left\langle \frac{\det(x-X')\det(y-Y')}{\det(y+Y')} \right\rangle' e^{-N\left(\frac{1-c}{2}x^2+\frac{1+c}{2}y^2-\frac{\hat{g}}{3}(x^3+3xy^2)\right)} \\
&\equiv N \int dxdy e^{-NV_{\text{eff}}(x,y)}.
\end{aligned}
$$

In the large-N limit

$$
\begin{aligned}
V_{\text{eff}}^{(0)}(x,y) &= \frac{1-c}{2}x^2 + \frac{1+c}{2}y^2 - \frac{\hat{g}}{3}(x^3+3xy^2) \\
&\quad -\frac{1}{N}\langle \operatorname{tr}\log(x-X)\rangle - \frac{1}{N}\langle \operatorname{tr}\log(y-Y)\rangle + \frac{1}{N}\langle \operatorname{tr}\log(y+Y)\rangle \\
&= \frac{1-c}{2}x^2 + \frac{1+c}{2}y^2 - \frac{\hat{g}}{3}(x^3+3xy^2) - \frac{1}{N}\langle \operatorname{tr}\log(x-X)\rangle .
\end{aligned}
$$

saddle point eq.:

$$\begin{aligned} 0 &= (1-c)x - \hat{g}(x^2+y^2) - \left\langle \frac{1}{N}\mathrm{tr}\frac{1}{x-X} \right\rangle, \\ 0 &= (1+c-2\hat{g}x)y. \end{aligned}$$

two exact solutions :

$$x_0 = \frac{1+c}{2\hat{g}}, \quad y_0 = \pm\frac{2c}{\hat{g}}$$

give two degenerate nonperturbative effects

$$V_{\mathrm{eff}}(x_0, y_0) = \frac{4\sqrt{6}}{\sqrt{7}g_s},$$

which agree with the two degenerate nonperturbative effects in the original model. → T-dual model misses the remaining one?
In fact, we miss a possibility that

$$x_0 \notin I_X, \quad y_0 \in I_Y \rightarrow (x_0, y_0) \notin \mathcal{S}.$$

In fact, by symmetry, we expect a saddle point with $y_0 = 0 \in I_Y$. In this case, we cannot expand $e^{-V_{\mathrm{eff}}}$ as above and cannot trust the above expression for $V_{\mathrm{eff}}^{(0)}$. Note that in the original model all saddle points are outside the cut for both coordinates.

Resolution: $O(n)$ model description

$y_0 \in I_Y \longrightarrow$ we can perform Y-integration first.

$$Z = \int dX \exp(-\Gamma)$$

$$\Gamma = N \mathrm{tr}\left(\frac{1-c}{2}X^2 - \frac{\hat{g}}{3}X^3\right) + \frac{1}{2}\mathrm{tr}\log\left(1\otimes 1 - \frac{1}{1+c}(X\otimes 1 + 1\otimes X)\right)$$

From this "one-matrix model", we can identify the instanton as an isolated eigenvalue. Then we find the remaining nonperturbative effect with $x_0 \notin I_X$:

Discussion: universality of nonperturbative effect including T-duality

Free energies of the original model and the dual model are the same, but we have to take the double scaling limit for each model.

$$F = \frac{C_0}{g_s^2} t^{7/3} + C_1 \log t + \cdots + \underbrace{D e^{-C/g_s}}_{\text{nonperturbative effect}}$$

String equation fixes the values of C_0, C_1, C, but not D.
In higher genus, T-duality is violated: the dual model contains the global vector field along the nontrivial homology cycle, which cannot be interpreted as a spin configuration. Asatani, T.K., Okawa, Sugino and Yoneya

$\longrightarrow$ difference of cylinder amplitudes

$\longrightarrow$ difference of D?

$\longrightarrow$ Universality of nonperturbative effect including T-duality

$\longrightarrow$ Universality of string theory itself including T-duality