" Joint Meeting of Pacific Region Particle Physics Communities "

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Amoebas and Instantons

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Talk based on

Takashi Maeda and T.N., "Amoebas and Instantons," hep-th/0601233.

T. Maeda, Y. Noma, T. Tamakoshi and T.N. , hep-th/0505083.

T. Maeda, K. Takasaki, T. Tamakoshi and T.N. , hep-th/0412329.

T. Maeda, K. Takasaki, T. Tamakoshi and T.N. , hep-th/0412327.

1. Viro's Dequantization and Toropical Geometry

- Maslov's dequantization of deformation quantizations says :

Tropical algebra $(\mathbb{R}, \max, +)$ **is a classical semiring of** $(\mathbb{R}_{>0}, +, \cdot)$

$$\frac{1}{R}\log() \ : \ \mathbb{R}_{>0} \longrightarrow \mathbb{R} \qquad x \longmapsto \frac{1}{R}\log x = u$$

$R > 0$ deformation parameter

$$(\mathbb{R}_{>0}, +, \cdot) \xrightarrow{\frac{1}{R}\log} (\mathbb{R}, \oplus_R, \odot_R) \overset{R\to\infty}{\Longrightarrow} (\mathbb{R}, \max, +)$$

$$\begin{cases} u \oplus_R v = \frac{1}{R}\log(e^{Ru} + e^{Rv}) \\ u \odot_R v = u + v \end{cases} \qquad \begin{cases} \oplus_\infty = \max \\ \odot_\infty = + \end{cases}$$

- **Tropical geometry is a piecewise linear (PL) geometry**

$$\textit{Tropical polynomials} \quad \text{“} \sum_i a_i u^i \text{”} \ = \ \max_i (a_i + iu)$$

(Use $\oplus_\infty$ for addition and $\odot_\infty$ for multiplication)

It is a dequantization of real algebraic geometry

$$\begin{matrix} \text{real curves} & & \textit{tropical curves} \\ y = \sum_i e^{Ra_i} x^i & \overset{\lim_{R\to\infty}\frac{1}{R}\log}{\Longrightarrow} & v = \max_i (a_i + iu) \end{matrix}$$

- **We see many examples of tropical geometry in gauge theory and string theory.**

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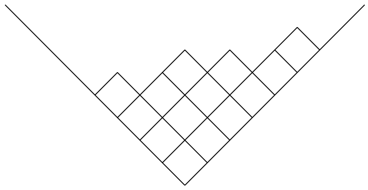
2. $5d$ $\mathcal{N}=1$ SUSY Gauge Theories

- 4d Nekrasov's functions are described by using Young diagrams (partitions).



Young diagram μ

$|\mu|$ = the number of boxes

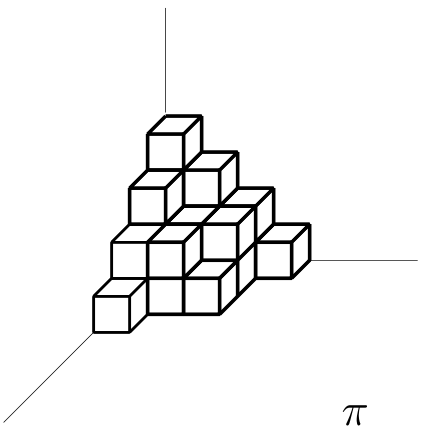
$$Z_{U(1)}(\Lambda,\,\hbar) \;=\; \sum_{\mu} \left(\frac{\Lambda}{\hbar}\right)^{2|\mu|} \frac{1}{\left(\prod_{(i,j)\in\mu} h(i,j)\right)^2}\,.$$

- 5d Nekrasov's functions are q-deformations and are described by using $3d$ Young diagrams (plane partitions).

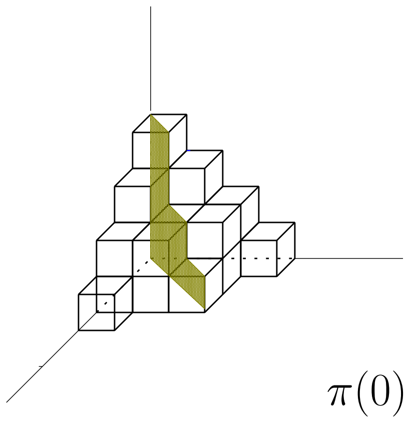
$$Z_{U(1)}(q,\,Q) \;=\; \sum_{\text{partitions } \mu} Q^{|\mu|} s_{\mu}(q)^2$$

$$\begin{cases} q = e^{-R\hbar}, & R \text{ : radius of } S^1 \\ Q = (R\Lambda)^2 \end{cases}$$

$$= \sum_{\text{plane partitions } \pi} q^{|\pi|} Q^{|\pi(0)|}$$



π

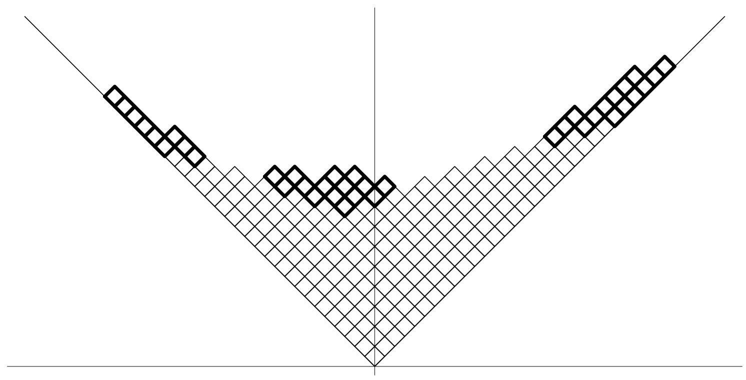


$\pi(0)$

- $5d$ Nekrasov's function of $SU(N)$ Yang-Mills is obtained from $Z_{U(1)}(q,Q)$.

Core and Quotients

$$\mu \overset{1:1}{\longleftrightarrow} (\lambda^{(1)},p_1),\cdots,(\lambda^{(N)},p_N) \overset{1:1}{\longleftrightarrow} \begin{array}{l} N\text{-core } (p_1,\cdots,p_N) \\ N\text{-quotients } (\lambda^{(1)},\cdots,\lambda^{(N)}) \end{array}$$



3 clusters of bold boxes correspond to 3-quotients $\lambda^{(1,2,3)}$.

3-core is of thin boxes.

Factorization by N-Cores

$$Z_{U(1)}(q,Q) = \sum_{\text{integers } p_1,\cdots,p_N} Z_{SU(N)}(p_1,\cdots,p_N;q,Q)$$

This factorization turns to be (hep-th/0412327)

$$Z_{SU(N)}(\{p_r\};q,Q) \;=\; Z_{5d\,\mathrm{SYM}}(\{a_r\};\Lambda,R,\hbar)$$

by identifying

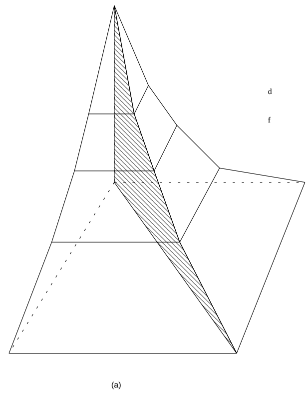
$$q = e^{-\frac{R}{N}\hbar}, \quad Q = (R\Lambda)^2, \quad p_r = a_r/\hbar\,.$$

The RHS is Nekrasov's function of $5d\,\mathcal{N}=1$ SUSY $SU(N)$ Yang-Mills + the Chern-Simons term.

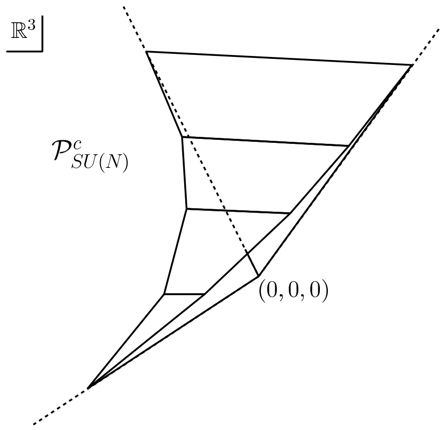
3. Ground State Crystal and Local $SU(N)$ Geometry

$$1/R \;=\; \text{Temp. in Statistical Model}$$

As $R \to \infty$ the ground state dominates. The ground state is the $3d$ Young diagram UNIQUELY determined by N-core $(p_1, \cdots, p_N)$.



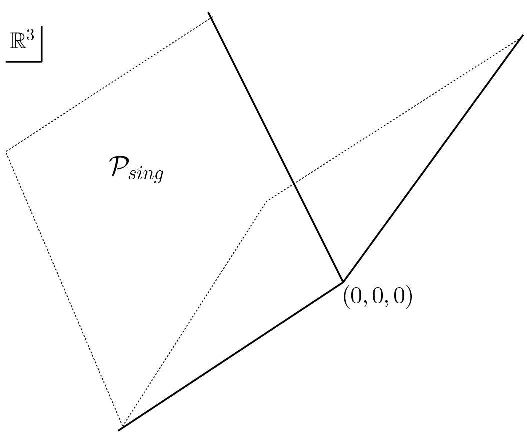
Ground state

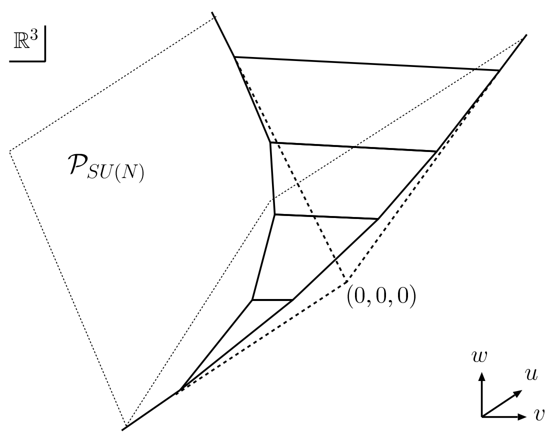


$\exists$ lattice homomorphism (degree N) that maps the ground state to a $3d$ solid $\mathcal{P}^c_{SU(N)}$.

$\mathcal{P}^c_{SU(N)}$ is the complement between two polyhedra $\mathcal{P}_{SU(N)}$ and $\mathcal{P}_{sing}$.

$$\mathcal{P}^c_{SU(N)} \;=\; \mathcal{P}_{sing} \setminus \mathcal{P}_{SU(N)}$$





- Two polyhedra descibe local geometries resposible to field theories

$\mathcal{P}_{sing}$	local geometry of $5d$ SCFT
$\mathcal{P}_{SU(N)}$	local geometry of $5d$ gauge theory

- The ground state energy becomes (hep-th/0505083)

$$E_{\text{ground state}} = \mathrm{Vol}\Big(\mathcal{P}^c_{SU(N)}\Big) = \frac{1}{6}\sum_{r>s}^{N}(a_r - a_s)^3 + \frac{N}{6}\sum_{r=1}^{N} a_r^3$$

<u>Comments</u>

$i)$ This is consistent with Vafa's "Quantum Calabi-Yau".

$ii)$ The volume can be a regularized Kahler volume of local geometry. (Noma's talk.)

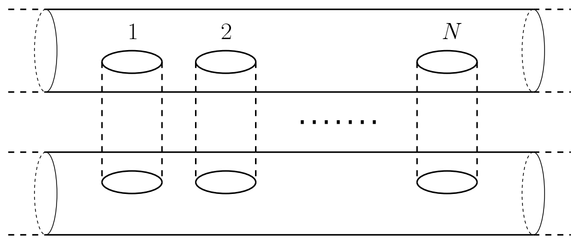
4. **Amoeba Degeneration and Crystal**

- Amoebas are mathematical ones (Gelfand-Kapranov-Zelevinsky)

<u>$5d$ version of $SU(N)$ SW curve</u>

$$V_{f_{SU(N)}} = \left\{ f_{SU(N)}(x,y) = 0 \in (\mathbb{C}^*)^2 \right\},$$

$$f_{SU(N)}(x,y) = \prod_{r=1}^{N} (x - \beta_r) - (R\Lambda)^N \big(y + y^{-1}\big)$$



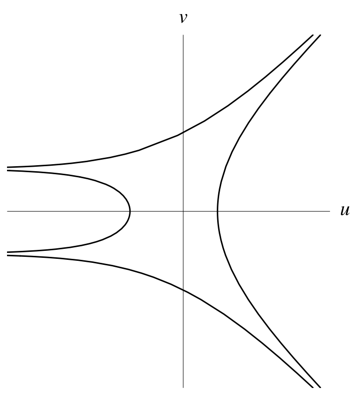
<u>Amoeba of the SW curve</u>

$$\mathcal{A}_{f_{SU(N)}} = \mathrm{Log}(V_{f_{SU(N)}}),$$

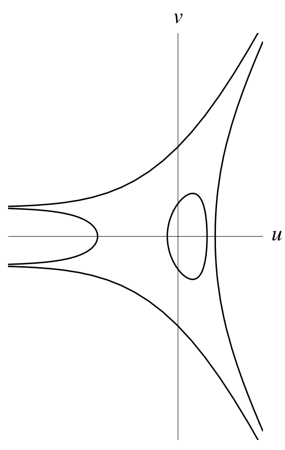
where $\quad \mathrm{Log} \; : \; (\mathbb{C}^*)^2 \longrightarrow \mathbb{R}^2$

$$(x,\, y) \longmapsto (\,\log|x|,\, \log|y|\,)\,.$$

The amoeba spreads 4 tentackles and has N-1 holes.

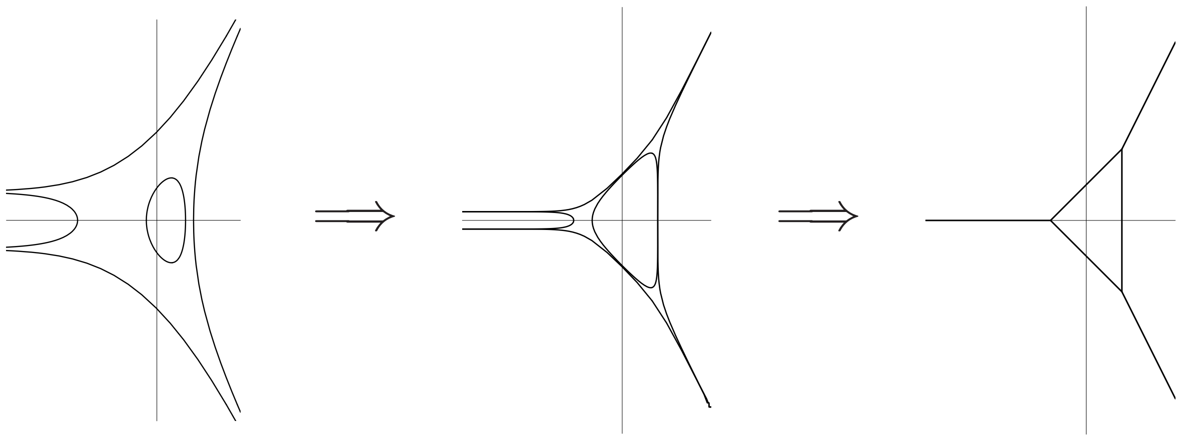


$N = 1$



$N = 2$

- As $R \to \infty$ the amoeba degenerates and realizes a tropical curve (essentially, by Viro's dequantization).

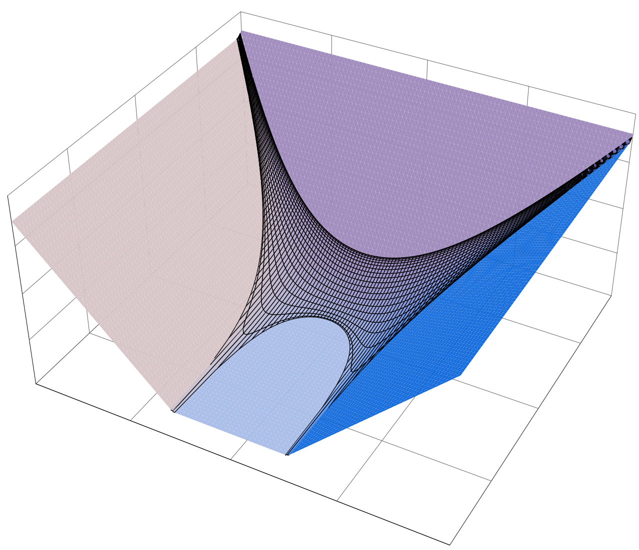


Use of Ronkin's function

$$N_{f_{SU(N)}}(\ ,\ ) \quad : \qquad \mathbb{R}^2 \longmapsto \mathbb{R} \cup \{\infty\}$$

$$N_{f_{SU(N)}}(u,v) = \frac{1}{R}\frac{1}{(2\pi i)^2}\int_{\substack{|x|=\exp Ru \\ |y|=\exp Rv}} \frac{dx}{x}\frac{dy}{y} \log \left| f_{SU(N)}(x,y) \right|$$

Ronkin's function is a convex function that is $i)$ strictly convex over the amoeba and $ii)$ linear over each connected component over the amoeba complement.

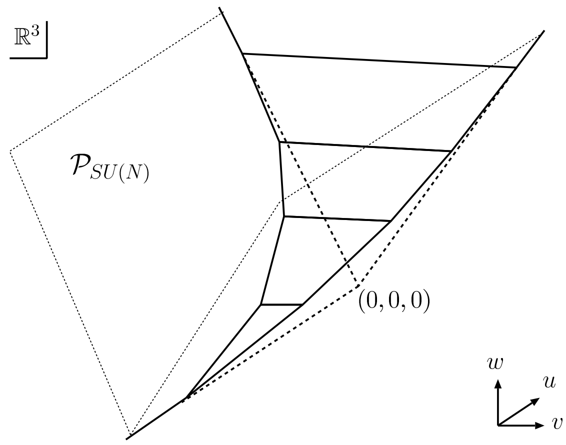


$N = 1$

The limit of Ronkin's function is the facet of $3d$ polyhedron $\mathcal{P}_{SU(N)}$. (hep-th/0601233)

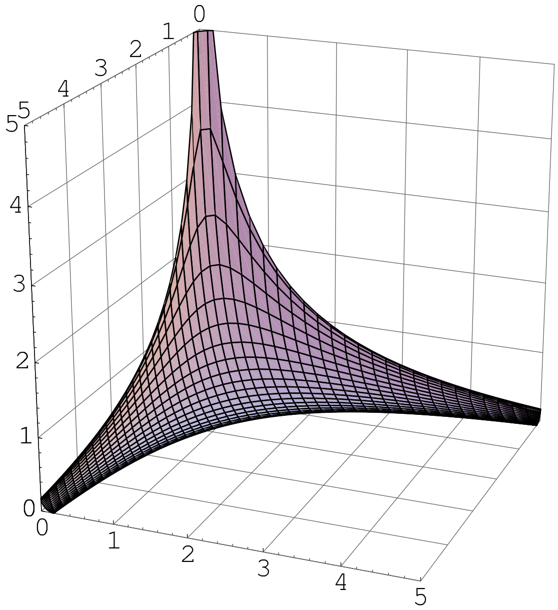
$$\left\{ (u, v, \lim_{R\to\infty} N_{f_{SU(N)}}(u,v)) \right\}$$

$$= \partial \mathcal{P}_{SU(N)}$$



5. Limit Shape and Ronkin's Function

- At finite R the ground state crystal (local geometry) is deformed by thermal excitation.



The limit shape realized in $Q = 1$ $U(1)$ model is interpreted by using Ronkin's function of $f(x, y) = 1 + x + y$. (Okounkov-Reshetikhin).

Limit shape $P_{\star}^{SU(N)}$ of the main diagonal Young diagram μ is the following section of the Ronkin function (hep-th/0601233)

$$N_{f_{SU(N)}}(u, 0) = \frac{N}{2}\Big(P_{\star}^{SU(N)}(u) + u\Big)$$

6. Summary

We have seen the correspondence

$\mathcal{N}=2$ SUSY gauge theories/$\mathbb{R}^4\times S^1$		Kähler gravities/local geometries
The Seiberg-Witten curves	$\Longleftrightarrow$	Limit shapes
$V_{f_{SU(N)}}$		$\mathbf{P}_\star^{SU(N)}$

through

♠ The Ronkin functions

$$N_{f_{SU(N)}}(u,0) \;=\; \frac{N}{2}\Big(P_\star^{SU(N)}(u)+u\Big)$$

♠ Tropical degeneration of the amoebas

$$\mathrm{Vol}\Big(\mathcal{P}^c_{SU(N)}\Big) = E_{\text{ground state}}$$

This issue will be tied up with the following perspectives

i) Integrable structure of the gauge theories

ii) The mirror symmetry

iii) Dimers, Algae, ...