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Amoebas and Instantons

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Talk based on


1. Viro's Dequantization and Toropical Geometry

• Maslov’s dequantization of deformation quantizations says:

**Tropical algebra** \((\mathbb{R}, \max, +)\) is a classical semiring of \((\mathbb{R}_{>0}, +, \cdot)\)

\[
\frac{1}{R} \log() : \mathbb{R}_{>0} \longrightarrow \mathbb{R} \quad x \mapsto \frac{1}{R} \log x = u
\]

\(R > 0\) deformation parameter

\[
(\mathbb{R}_{>0}, +, \cdot) \overset{\frac{1}{R} \log}{\longrightarrow} (\mathbb{R}, \oplus_R, \odot_R) \overset{R \to \infty}{\longrightarrow} (\mathbb{R}, \max, +)
\]

\[
\left\{
\begin{array}{l}
    u \oplus_R v = \frac{1}{R} \log(e^{Ru} + e^{Rv}) \\
    u \odot_R v = u + v
\end{array}
\right.
\]

\[
\left\{
\begin{array}{l}
    \oplus_\infty = \max \\
    \odot_\infty = +
\end{array}
\right.
\]

• Tropical geometry is a piecewise linear (PL) geometry

**Tropical polynomials** " \(\sum_i a_i u^i \) " = \(\max_i (a_i + iu)\)

(Use \(\oplus_\infty\) for addition and \(\odot_\infty\) for multiplication)

It is a dequantization of real algebraic geometry

real curves \(\lim_{R \to \infty} \frac{1}{R} \log\) tropical curves

\[
y = \sum_i e^{Ra_i x^i} \quad v = \max_i (a_i + iu)
\]

• We see many examples of tropical geometry in gauge theory and string theory.
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2. $5d \mathcal{N} = 1$ SUSY Gauge Theories

- 4d Nekrasov’s functions are described by using Young diagrams (partitions).

  \[ Z_{U(1)}(\Lambda, \hbar) = \sum_{\mu} \left( \frac{\Lambda}{\hbar} \right)^{2|\mu|} \frac{1}{(\prod_{(i,j) \in \mu} h(i,j))^{2}} \cdot \]

  - 5d Nekrasov’s functions are $q$-deformations and are described by using 3d Young diagrams (plane partitions).

  \[ Z_{U(1)}(q, Q) = \sum_{\text{partitions } \mu} Q^{\mu} s_{\mu}(q)^{2} \]

  \[ = \sum_{\text{plane partitions } \pi} q^{\pi} Q^{\pi(0)} \]

\[ \frac{1}{(\prod_{(i,j) \in \mu} h(i,j))^{2}}. \]
• 5d Nekrasov’s function of $SU(N)$ Yang-Mills is obtained from $Z_{U(1)}(q, Q)$.

Core and Quotients

\[
\mu \overset{1:1}{\leftrightarrow} (\lambda^{(1)}, p_1), \ldots, (\lambda^{(N)}, p_N) \overset{1:1}{\leftrightarrow} N\text{-core } (p_1, \ldots, p_N) \quad N\text{-quotients } (\lambda^{(1)}, \ldots, \lambda^{(N)})
\]

3 clusters of bold boxes correspond to 3-quotients $\lambda^{(1,2,3)}$.

3-core is of thin boxes.

Factorization by N-Cores

\[
Z_{U(1)}(q, Q) = \sum_{\text{integers } p_1, \ldots, p_N} Z_{SU(N)}(p_1, \ldots, p_N; q, Q)
\]

This factorization turns to be (hep-th/0412327)

\[
Z_{SU(N)}(\{p_r\}; q, Q) = Z_{5d\text{SYM}}(\{a_r\}; \Lambda, R, \hbar)
\]

by identifying

\[
q = e^{-\frac{R}{N}\hbar}, \quad Q = (R\Lambda)^2, \quad p_r = a_r/\hbar.
\]

The RHS is Nekrasov’s function of 5d $\mathcal{N} = 1$ SUSY $SU(N)$ Yang-Mills + the Chern-Simons term.
3. Ground State Crystal and Local $SU(N)$ Geometry

\[ \frac{1}{R} = \text{Temp. in Statistical Model} \]

As $R \to \infty$ the ground state dominates.

The ground state is the $3d$ Young diagram UNIQUELY determined by $N$-core $(p_1, \cdots, p_N)$.

\[ \exists \text{ lattice homomorphism (degree } N \text{) that maps the ground state to a } 3d \text{ solid } \mathcal{P}^c_{SU(N)}. \]

$\mathcal{P}^c_{SU(N)}$ is the complement between two polyhedra $\mathcal{P}_{SU(N)}$ and $\mathcal{P}_{sing}$.

\[ \mathcal{P}^c_{SU(N)} = \mathcal{P}_{sing} \setminus \mathcal{P}_{SU(N)} \]
• Two polyhedra describe local geometries responsible for field theories

\[ \mathcal{P}_{\text{sing}} \quad \text{local geometry of } 5d \text{ SCFT} \]
\[ \mathcal{P}_{SU(N)} \quad \text{local geometry of } 5d \text{ gauge theory} \]

• The ground state energy becomes (hep-th/0505083)

\[
E_{\text{ground state}} = \text{Vol}\left(\mathcal{P}_{SU(N)}^c\right) = \frac{1}{6} \sum_{r>s}^N (a_r - a_s)^3 + \frac{N}{6} \sum_{r=1}^N a_r^3
\]

Comments

i) This is consistent with Vafa’s “Quantum Calabi-Yau”.

ii) The volume can be a regularized Kahler volume of local geometry. (Noma’s talk.)
4. Amoeba Degeneration and Crystal

- Amoebas are mathematical ones (Gelfand-Kapranov-Zelevinsky)

**5d version of SU(N) SW curve**

\[
V_{f_{SU(N)}} = \left\{ f_{SU(N)}(x, y) = 0 \in (\mathbb{C}^*)^2 \right\},
\]

\[
f_{SU(N)}(x, y) = \prod_{r=1}^{N} (x - \beta_r) - (RA)^N(y + y^{-1})
\]

Amoeba of the SW curve

\[
A_{f_{SU(N)}} = \text{Log}(V_{f_{SU(N)}}),
\]

where \( \text{Log} : (\mathbb{C}^*)^2 \rightarrow \mathbb{R}^2 \)

\[(x, y) \mapsto (\log |x|, \log |y|).\]

The amoeba spreads 4 tentacles and has N-1 holes.
• As $R \to \infty$ the amoeba degenerates and realizes a tropical curve (essentially, by Viro’s dequantization).

Use of Ronkin’s function

$$N_{f_{SU(N)}}(u, v) : \mathbb{R}^2 \mapsto \mathbb{R} \cup \{\infty\}$$

$$N_{f_{SU(N)}}(u, v) = \frac{1}{R} \frac{1}{(2\pi i)^2} \int_{|x|=\exp Ru \atop |y|=\exp Rv} \frac{dx \, dy}{x} \log |f_{SU(N)}(x, y)|$$

Ronkin’s function is a convex function that is $i$) strictly convex over the amoeba and $ii$) linear over each connected component over the amoeba complement.

$$N = 1$$
The limit of Ronkin’s function is the facet of 3d polyhedron $\mathcal{P}_{SU(N)}$. (hep-th/0601233)

$$\left\{ (u, v, \lim_{R \to \infty} N_{f_{SU(N)}}(u, v)) \right\} = \partial \mathcal{P}_{SU(N)}$$
5. Limit Shape and Ronkin’s Function

- At finite $R$ the ground state crystal (local geometry) is deformed by thermal excitation.

The limit shape realized in $Q = 1$ $U(1)$ model is interpreted by using Ronkin’s function of $f(x, y) = 1 + x + y$. (Okounkov-Reshetikhin).

Limit shape $P^{SU(N)}_\star$ of the main diagonal Young diagram $\mu$ is the following section of the Ronkin function (hep-th/0601233)

$$N_{f^{SU(N)}}(u, 0) = \frac{N}{2} \left( P^{SU(N)}_\star(u) + u \right)$$
6. Summary

We have seen the correspondence

\[ \mathcal{N} = 2 \text{ SUSY gauge theories}/\mathbb{R}^4 \times S^1 \quad \text{Kähler gravities/local geometries} \]

The Seiberg-Witten curves \( V_{f_{SU(N)}} \) ☇ Limit shapes \( P^*_{SU(N)} \)

through

♣ The Ronkin functions

\[ N_{f_{SU(N)}}(u, 0) = \frac{N}{2} \left( P^*_{SU(N)}(u) + u \right) \]

♣ Tropical degeneration of the amoebas

\[ \text{Vol}(P^e_{SU(N)}) = E_{\text{ground state}} \]

This issue will be tied up with the following perspectives

i) Integrable structure of the gauge theories

ii) The mirror symmetry

iii) Dimers, Algae, ...