On Higher Derivative Terms in Effective Action on Soliton

Toshiaki Fujimori (Tokyo institute of Technology)

Kaneyasu Asakuma, Minoru Eto, Muneto Nitta, Keisuke Ohashi, Norisuke Sakai

1. Introduction

effective theory on solitons

$$\mathcal{L}_{\rm eff} = g_{i\bar{j}}(\phi, \bar{\phi})\partial_\mu \phi^i \partial^\mu \bar{\phi}^j$$ · · · **2-derivative terms**

higher derivative terms

position moduli

effective action → $\mathcal{L}_{\rm eff} = -T\sqrt{-\det\left(\partial_\mu X^\alpha \partial_\nu X_\alpha\right)}$

· · · **Nambu-Goto action**

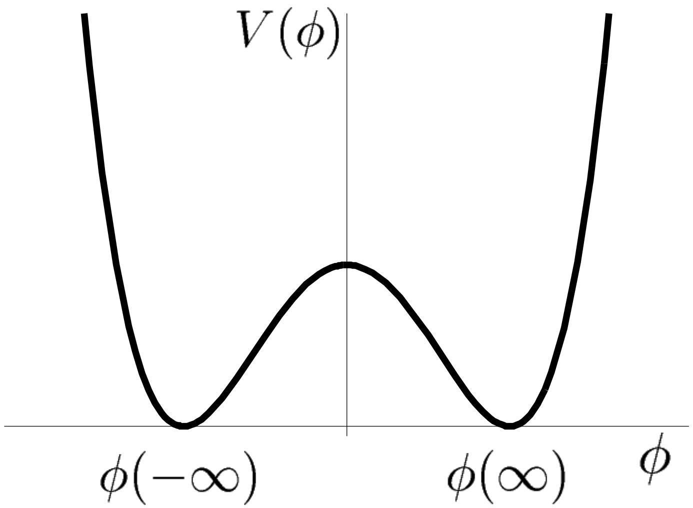
internal moduli { **phase** — **domain wall**; **CP^{N-1}** — **vortex** }

→ **higher derivative terms**

2. Method

Simplest model

$$\mathcal{L} = -\frac{1}{2}\partial_M\phi\partial^M\phi - V(\phi)$$



equation of motion

$$-\partial_M\partial^M\phi + \frac{\partial V}{\partial\phi} = 0$$

wall configuration
- **depends only** y **(coordinate of co-dim)**
- **interpolates** $\phi(-\infty)$ **and** $\phi(\infty)$

$\phi_{\rm sol}(y)$	$\longrightarrow$	$\phi_{\rm sol}(y-Y)$
Wall position		**Wall position**
$y=0$		$y=Y$

Y **: moduli parameter**

Effective theory

Promote the moduli parameters to fields which depend on the worldvolume coordinates

$$Y \to Y(x)$$ **moduli parameter → moduli field**

Dependence on worldvolume coordinates is assumed to be small

$$\partial_\mu \sim \lambda$$ λ **: small parameter**

λ - expansion

$$\phi = \phi_{\rm sol}(y - Y(x)) + \sum_k \phi^{(k)}$$

$$S = \sum_k S^{(k)}$$

$$\phi^{(k)},\ S^{(k)} \sim \mathcal{O}(\lambda^k)$$

$S_{\text{eff}}^{(0)}$ and $S_{\text{eff}}^{(2)}$ can be obtained by substituting the solution $\phi_{\text{sol}}(y - Y(x))$ into the original action

$$S_{\text{eff}}^{(0)} + S_{\text{eff}}^{(2)} = S[\phi_{\text{sol}}(y - Y(x))]$$

$$= \int d^{d-1}x \Big[\underbrace{-\partial_y \phi_{\text{sol}} \partial^y \phi_{\text{sol}} - V(\phi_{\text{sol}})}_{\mathcal{O}(\lambda^0)} \underbrace{- \partial_\mu \phi_{\text{sol}} \partial^\mu \phi_{\text{sol}}}_{\mathcal{O}(\lambda^2)} \Big]$$

$$= \int d^{d-1}x \Big[\underbrace{-T}_{\mathcal{O}(\lambda^0)} \underbrace{- \frac{T}{2} \partial_\mu Y \partial^\mu Y}_{\mathcal{O}(\lambda^2)} \Big]$$

$\longrightarrow$ $\partial_\mu \partial^\mu Y = 0$ **up to** $\mathcal{O}(\lambda^2)$

$\mathcal{O}(\lambda^2)$ equation of motion

$$\Delta\phi^{(2)} - \partial_\mu\partial^\mu\phi_{\rm sol} = 0 \qquad \Delta = -\partial_y^2 + \frac{\partial^2 V}{\partial\phi^2}$$

$\partial_Y\phi_{\rm sol}$: **zero mode** of operator Δ

inner product

$$\langle f_1,\, f_2\rangle = \int dy\, f_1 f_2 \qquad f_1,\, f_2 \text{ : function of } y$$

orthogonal decomposition

$$\Delta\phi^{(2)} - \partial_\mu\partial^\mu\phi_{\rm sol} = 0$$

→ $$\Delta\phi^{(2)} - \partial_\mu Y\partial^\mu Y\partial_Y^2\phi_{\rm sol} = 0$$

→ $$-\partial_\mu\partial^\mu Y\partial_Y\phi_{\rm sol} = 0 \quad \cdots\ (*)$$

$(*)$ is equivalent to the equation of motion obtained from $S_{\rm eff}^{(2)}$

solution of the $\mathcal{O}(\lambda^2)$ equation of motion

$$\phi^{(2)} = \frac{1}{2}\partial_\mu Y \partial^\mu Y (y - Y) \partial_Y \phi_{\rm sol}$$

$\mathcal{O}(\lambda^4)$ effective action

$$S^{(4)} = \int d^{d-1}x \, \frac{1}{2} \left\langle \phi^{(2)}, \, \Delta\phi^{(2)} \right\rangle = \int d^{d-1}x \, \frac{T}{8} \left(\partial_\mu Y \partial^\mu Y\right)^2$$

$$S_{\rm eff} = -T \int d^{d-1}x \left[1 + \frac{1}{2}\partial_\mu Y \partial^\mu Y - \frac{1}{8}\left(\partial_\mu Y \partial^\mu Y\right)^2 + \mathcal{O}(\lambda^6)\right]$$

This effective action coincide with the expansion of Nambu-Goto action

$$S_{\rm NG} = -T \int d^{d-1}x \, \sqrt{-\det\left(\partial_\mu X^M \partial_\nu X_M\right)}$$

$$= -T \int d^{d-1}x \left[1 + \frac{1}{2}\partial_\mu Y \partial^\mu Y - \frac{1}{8}\left(\partial_\mu Y \partial^\mu Y\right)^2 + \cdots\right]$$

3. BPS domain wall in CP^1 sigma model

action

$$S = \int d^d x \, \left[-g_{\phi\bar{\phi}} \partial_M \phi \partial^M \bar{\phi} - g_{\phi\bar{\phi}} k \bar{k} \right], \quad g_{\phi\bar{\phi}} = \frac{c}{(1+|\phi|^2)^2}, \quad k = im\phi$$

vacua

$$\phi = 0, \ \infty$$ → **domain wall**

BPS equation

$$\partial_y \phi - m\phi = 0$$

solution (BPS domain wall)

$$\phi_{\rm sol} = e^{m(y-Y+i\theta)}$$

moduli parameters Y : position , θ : phase

$$S_{\rm eff}^{(0)} + S_{\rm eff}^{(2)} = S[\phi_{\rm sol}]$$

$$= -T \int d^{d-1}x \left(1 + \frac{1}{2} \partial_\mu Y \partial^\mu Y + \frac{1}{2} \partial_\mu \theta \partial^\mu \theta \right)$$

(2+1)-d worldvolume

$$*d\theta = F$$

U(1) gauge field

$\mathcal{O}(\lambda^2)$ equation of motion

$$V = g_{\phi\bar{\phi}} k\bar{k}$$

$$\Delta\phi^{(2)} - \mathcal{D}_\mu \partial^\mu \phi_{\rm sol} = 0\,, \qquad \Delta = \mathcal{D}_y^2 - R^\phi{}_{\phi\bar{\phi}\phi}\partial_y\phi\partial_y\bar{\phi} + \nabla_\phi\partial_\phi V$$

$\mathcal{D}_\mu,\ \mathcal{D}_y$ **: pullback of covariant derivative on target space**

The operator Δ can be written as

$$\Delta = D^\dagger D\,, \qquad D = \mathcal{D}_y + i\nabla_\phi k\,, \quad D^\dagger = -\mathcal{D}_y + i\nabla_\phi k\,,$$

$D^\dagger$ is adjoint of D

$$\langle Df_1,\ f_2\rangle = \langle f_1,\ D^\dagger f_2\rangle$$

$$\langle f_1,\ f_2\rangle = \int dy\, g_{\phi\bar{\phi}} f_1 \overline{f_2}$$

$$S_{\rm eff}^{(4)} = \int d^{d-1}x\, \frac{1}{2}\left\langle \phi^{(2)},\ \Delta\phi^{(2)}\right\rangle = \int d^{d-1}x\, \frac{1}{2}\left\langle D\phi^{(2)},\ D\phi^{(2)}\right\rangle$$

equation of motion for $D\phi^{(2)}\ldots$ **first order differential eq.**

$$D^{\dagger}\left(D\phi^{(2)}\right)-\mathcal{D}_{\mu}\partial^{\mu}\phi_{\rm sol}=0$$ ⟹ **solution** $D\phi^{(2)}$

$$S_{\rm eff}^{(4)}=\int d^{d-1}x\,\langle D\phi^{(2)},\,D\phi^{(2)}\rangle$$
$$=\frac{T}{8}\int d^{d-1}x\left[\Big(\partial_{\mu}Y\partial^{\mu}Y-\partial_{\mu}\theta\partial^{\mu}\theta\Big)^{2}+4\Big(\partial_{\mu}Y\partial^{\mu}\theta\Big)^{2}\right]$$

The effective action coincide with expansion of the following action:

$$S=-T\sqrt{-\det\left(\eta_{\mu\nu}+\partial_{\mu}Y\partial_{\nu}Y+\partial_{\mu}\theta\partial_{\nu}\theta\right)}$$

In the case of (2+1)-dimensional worldvolume this action can be dualized into the DBI action

4. BPS vortex in U(N) gauge theory

action $U(N)$ **gauge field** $+$ N **fundamental scalars**

$$S = \int d^d x \, \mathrm{Tr}\left[-\frac{1}{2g^2} F_{MN} F^{MN} + \mathcal{D}_M H \left(\mathcal{D}^M H\right)^\dagger - \frac{g^2}{4}\left(H H^\dagger - c\mathbf{1}_{N_C}\right)^2\right]$$

H : N **fundamental scalars**, $N \times N$ **matrix**

F_{MN} : **field strength**, W_M : **gauge field**

g : **gauge coupling**, c : **FI parameter**

vacuum

$$H = \begin{pmatrix} \sqrt{c} & & \\ & \ddots & \\ & & \sqrt{c} \end{pmatrix}$$

BPS equations $z = x^1 + ix^2$ **: complex coordinate**

$$F_{z\bar{z}} + i\frac{g^2}{4}(c\mathbf{1}_{N_C} - HH^\dagger) = 0, \qquad \mathcal{D}_{\bar{z}}H = 0$$

one vortex solution $U(2)$ **gauge theory**

$$H_{\rm sol} = U^\dagger \begin{pmatrix} H_{U(1)} & 0 \\ 0 & \sqrt{c} \end{pmatrix} U, \quad (F_{z\bar{z}})_{\rm sol} = U^\dagger \begin{pmatrix} F_{U(1)} & 0 \\ 0 & 0 \end{pmatrix} U$$

$$U = \begin{pmatrix} \phi_1 & \phi_2 \\ -\phi_2^* & \phi_1^* \end{pmatrix} \in SU(2)$$

$H_{U(1)}$, $F_{U(1)}$ **: one vortex solution in** $U(1)$ **gauge theory**

$$SU(2)/U(1) = \mathbf{C}P^1$$

$b = -\dfrac{\phi_2^*}{\phi_1^*}$ **: inhomogeneous coordinate of** $\mathbf{C}P^1$

gauge field (worldvolume direction)

$$-\mathcal{D}^\alpha F_{\alpha\mu} + i\frac{g^2}{2}\left(H(\mathcal{D}_\mu H)^\dagger - (\mathcal{D}_\mu H)H^\dagger\right) = 0$$

→ **solution** $(W_\mu)_{\rm sol}$

substituting $H_{\rm sol}$, $(W_z)_{\rm sol}$, $(W_\mu)_{\rm sol}$ **, we obtain**

$$S_{\rm eff}^{(2)} = -\frac{8\pi}{g^2}\int d^{d-2}x\, \frac{\partial_\mu b \partial^\mu \bar{b}}{(1+|b|^2)^2}$$

worldvolume $\cdots$ **(2+1)-d** $w = x^3 + ix^4$

½ BPS lump on ½ BPS vortex ⟺ **¼ BPS composite state of vortex and instanton**

$b = b(w)$ $\cdots$ **holomorphic map**

topological charge $\cdots$ $\frac{8\pi^2}{g^2}$ **= instanton charge**

$\mathcal{O}(\lambda^2)$ equation of motion

$$\Delta\mathbf{\Phi}^{(2)} - \mathcal{D}^\mu\mathbf{\Psi}_\mu = 0$$

$$\mathbf{\Phi}^{(2)} = \begin{pmatrix} W_z^{(2)} \\ W_{\bar{z}}^{(2)} \\ H^{(2)} \\ \left(H^{(2)}\right)^\dagger \end{pmatrix}, \qquad \mathcal{D}^\mu\mathbf{\Psi}_\mu = \begin{pmatrix} \mathcal{D}^\mu F_{\mu z} \\ \mathcal{D}^\mu F_{\mu\bar{z}} \\ \mathcal{D}_\mu\mathcal{D}^\mu H \\ \mathcal{D}_\mu\mathcal{D}^\mu H^\dagger \end{pmatrix}$$

The operator Δ is complicated

$$\Delta = \begin{pmatrix} -2\mathcal{D}_z\mathcal{D}_{\bar{z}} + g^2 HH^\dagger & 2\mathcal{D}_z\mathcal{D}_z & -i\frac{g^2}{2}\left(H^\dagger\right)^R\mathcal{D}_z & i\frac{g^2}{2}\left(H\mathcal{D}_z - \mathcal{D}_z H\right) \\ 2\mathcal{D}_{\bar{z}}\mathcal{D}_{\bar{z}} & -2\mathcal{D}_z\mathcal{D}_{\bar{z}} + g^2\left(HH^\dagger\right)^R & i\frac{g^2}{2}\left(\left(\mathcal{D}_{\bar{z}}H^\dagger\right)^R - \left(H^\dagger\right)^R\mathcal{D}_{\bar{z}}\right) & i\frac{g^2}{2}H\mathcal{D}_{\bar{z}} \\ -2iH^R\mathcal{D}_{\bar{z}} & -2i\left(2\left(\mathcal{D}_z H\right)^R + H^R\mathcal{D}_z\right) & -4\mathcal{D}_z\mathcal{D}_{\bar{z}} + \frac{g^2}{2}\left(H^\dagger H\right)^R & \frac{g^2}{2}HH^R \\ 2i\left(2\mathcal{D}_{\bar{z}}H^\dagger + H^\dagger\mathcal{D}_{\bar{z}}\right) & 2iH^\dagger\mathcal{D}_z & \frac{g^2}{2}H^\dagger\left(H^\dagger\right)^R & -4\mathcal{D}_{\bar{z}}\mathcal{D}_z + \frac{g^2}{2}H^\dagger H \end{pmatrix}$$

As in the previous case, the operator Δ can be written as

$$\Delta = D^\dagger D$$

$$D = \begin{pmatrix} -\mathcal{D}_{\bar{z}} & \mathcal{D}_z & -i\frac{g^2}{4}\left(H^\dagger\right)^R & -i\frac{g^2}{4}H \\ \mathcal{D}_{\bar{z}} & -\mathcal{D}_z & i\frac{g^2}{4}\left(H^\dagger\right)^R & i\frac{g^2}{4}H \\ 0 & 2iH^R & 2\mathcal{D}_{\bar{z}} & 0 \\ -2iH^\dagger & 0 & 0 & 2\mathcal{D}_z \end{pmatrix}, \quad D^\dagger = \begin{pmatrix} \mathcal{D}_z & -\mathcal{D}_z & 0 & i\frac{g^2}{2}H \\ -\mathcal{D}_{\bar{z}} & \mathcal{D}_{\bar{z}} & -i\frac{g^2}{2}\left(H^\dagger\right)^R & 0 \\ iH^R & -iH^R & -2\mathcal{D}_z & 0 \\ iH^\dagger & -iH^\dagger & 0 & -2\mathcal{D}_{\bar{z}} \end{pmatrix}$$

$$D^\dagger\left(D\Phi^{(2)}\right) - \mathcal{D}^\mu \Psi_\mu = 0 \cdots$$ **first order differential eq.**

→ **solution** $D\Phi^{(2)}$

$$S_{\text{eff}}^{(4)} = \int d^{d-2}x \left(\text{Tr}\left[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\mu}\right] + \frac{1}{2}\left\langle D\Phi^{(2)}, D\Phi^{(2)}\right\rangle\right)$$

$$= \frac{A}{g^4 c}\int d^{d-2}x \frac{|\partial_\mu b \partial^\mu b|^2}{(1+|b|^2)^4}$$

BPS configuration and topological charge are not affected

5. Conclusion

- Higer derivative terms in the effective actions on solitons are calculated.

- domain wall ･･･ position and phase
  In the case of (2+1)-d worldvolume, higher derivative terms coincide with that appearing in the expansion of DBI action.

- vortex ･･･ internal moduli (CP^1)
  ½ BPS soliton on vortex (instanton inside vortex) is not affected by 4-drivative term.