Towards a complete 4-dim Lagrangian form
Heterotic string theory

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Motivation

We are interested in compactifications of het. string theory to $N = 1, 4$-dim effective theories. In particular, we want to write an effective Lagrangian.

To find such 4-dim supersymmetric theories, one can the following:

1. Choose a space time $M^{10} = X \times M^4$, $M^4$ is 4-dim Mink. Take $X$ to be CY-space (complex Kaehler manifold with trivial canonical bundle)
   In particular, we want to find a Ricci-flat metric $g_{i\bar{j}}$ on $X$. If we fix a Kaehler class $H = [g_{i\bar{j}}]$, a theorem of Yau ensures the existence of a unique such metric.

2. Solve hermitian Yang-Mills equation on $X$, that is, find a gauge potential $A_i$, such that

$$F_{i\bar{j}}(A) = 0, \quad F_{i\bar{j}}^*(A) = 0, \quad g^{i\bar{j}} F_{i\bar{j}}(A) = 0.$$ 

Such $A_i$ can be interpreted as connection of holomorphic vector bundle $V$. 
Theorems of Donaldson-Uhlenbeck-Yau show, that if $V$ is a holomorphic bundle, which is stable with respect to $H$, then there exists a unique such solution $A_i$.

3. Find the field content of the 4-dim theory. It is determined by the zero modes $\psi^\alpha$ of the Dirac operator on $X$. This amounts to find harmonic differential forms $\psi$ valued in $V$, i.e. solutions of

$$0 = (\bar{\partial} + \bar{A})\psi = (\bar{\partial} + \bar{A})^* \psi,$$

where $*$ denotes the adjoint operator. These modes can be described algebraically as the sheaf cohomology of $V$.

4. Find 4-dim superpotential, that is, the wedge products of differential forms $\psi^\alpha$. This also can be done algebraically, e.g. rank three case, one needs to evaluate the map

$$H^1(X, V) \otimes H^1(X, V) \otimes H^1(X, V) \to H^3(X, \wedge^3 V) = \mathbb{C}.$$

5. Find the Kaehler potential for eff. 4-dim theory, or at least, the Kaehler metric which gives normalization for the 4-dim particles.
For example, let consider the normalization for the massless 4-dim fields which correspond to the moduli of the hermitian Yang-Mills gauge potential. Consider a deformation $\delta A$ of a solution $A_0$ to the HYM equation

$$A' = A_0 + \delta A.$$ 

That is, $A'$ solves the HYM equations as well. These deformations correspond to harmonic representatives of the cohomology classes of $H^1(X, V \otimes V^*)$ and give rise in 4-dim. to complex scalar fields $\phi$. After dimensional reduction of the 10-dim Yang-Mills action, one obtains

$$S_{\text{kin}}(\phi) = \sum \int_X d^6 \text{tr}(\delta p A_i, \delta \bar{q} A_{\bar{j}}) g^{i\bar{j}} \int_{M_4} d^4 x \partial_\mu \phi_p \partial^\mu \phi_{\bar{q}}$$

where $p, \bar{q}$ run over a basis of $H^1(X, V \otimes V^*)$ and denote the projections to the $(1, 0)$-form and $(0, 1)$-form part of $\delta A$ respectively. $g^{i\bar{j}}$ is the Ricci flat metric on $X$. 
In this talk I present our method on how to obtain numerical solutions to hermite Yang-Mills equation for stable bundles which can be applied on any compact Kaehler manifold.

I show how to apply this method to a stable rank three bundle $V$ on a quintic.

In addition, I present a method on how to compute the Kaehler metric

$$g_{p\bar{q}} = \int_X d^6 tr(\delta_p A_i, \delta_{\bar{q}} A_{\bar{j}}) g^{ij}$$

numerically. There are generalizations of this method that can be applied to obtain the normalization of all 4-dim fields.
A warm up, let's review the numerical computation of Ricci flat metrics on quintics, as described yesterday by Robert Karp. The basic idea is to consider embeddings of a quintic into a complex projective space of high dimension $N$

$$X \subset \mathbb{P}^N.$$ 

Projective spaces have a set a large set of Kaehler metrics, described by the Kaehler potentials

$$K_h = \log \left( \sum_{i\bar{j}} h^{i\bar{j}} Z_i \bar{Z}_j \right), \quad i\bar{j} = 0, \ldots N.$$ 

Note that the dimension of the family of Kaehler potential grows like $N^2$, hence for large $N$ there is a large such family. The idea is, as suggested by Yau, Tian, Luo and Donaldson, that the pull-back of the "correct" Kaehler metric given by $K_{hc}$ is a good approximation to the Ricci flat metric and agrees with it in the limit $N \to \infty$. 


More formally, the embedding is given the global sections $s_\alpha$ of the $k$ tensor power of the hyperplane bundle $\mathcal{O}_X(1)$ using the evaluation map

$$x \to (s_1(x), \ldots, s_N(x)) \in \mathbb{C}^N.$$ 

Donaldson showed recently that the "correct" Kaehler metric on $\mathbb{P}^N$ is given by the fixed point $T(h) = h$ of the $T$-operator

$$T(h)_{\alpha\bar{\beta}} \equiv \frac{N + 1}{vol(X)} \int_X dV \frac{s_\alpha \bar{s}_{\bar{\beta}}}{\sum h^{\alpha\bar{\beta}} s_\alpha \bar{s}_{\bar{\beta}}}.$$ 

and that $T^n(h)$ converges for $n \to \infty$. 

Solution to hermitian Yang-Mills equation

To apply this strategy to solve for the hermitian Yang-Mills connection $A_i$, recall that on a complex vector bundle $V$, there is a correspondence between a hermitian metric $(,): \bar{V} \otimes V \to \mathbb{C}^\infty(X)$, which in a frame $\{e_b\}$ gives $H_{a\bar{b}} = (e_a, e_b)$ and its corresponding metric connection $A_i = H^{-1} \partial_i H$.

Instead of solving for the hermitian Yang-Mills connection $A_i$, we solve for the hermite-Einstein metric $H$ in the gauge $\bar{A} = 0$.

Hence we want to find a large family of metrics on the holomorphic vector bundle $V$ (with fixed rank $r$). We consider the bundle $V \otimes \mathcal{O}_X(k)$ with $M$ global sections such that the map $X \to G(r, M)$ defines an embedding into the Grassmanian $G(r, M)$.

More precisely, after choosing a local frame for $V$, a basis of sections determines a $M \times r$ matrix $\{z_{a\alpha}\}_{a=1,\ldots,r, \alpha=1,\ldots,M}$. It is defined up to a $GL(M)$ change of basis and up to a $GL(r)$ change of frames, hence gives for each $x \in X$ we obtain a point in the Grassmanian $G(r, M)$. 
Now we get a set of natural metrics on $V(k)$

$$H = (zG^{-1}z^\dagger)^{-1}.$$

parameterized by $M \times M$ matrix $G$. Again, one wants to find a natural metric of this form which is a good approximation to hermite-Einstein metric and agrees with it in the large volume limit.

To find this metric, we define a generalized T-operator (DKLR)

$$T(G) = \frac{N}{Vr} \int_X z(z^*G^{-1}z)^{-1}z^*dV.$$

and conjecture that for any stable bundle $V$ it convergence to a fixed point $G_\infty$. We test this conjecture for $T\mathbb{P}^2$, a rank three bundle on $\mathbb{P}^2$ and rank three bundles on quintics. Such fixed point $G_\infty$ leads to a natural metric on $V$ for any $k$

$$H_k = H^{(k)} \otimes h^{-k},$$

where $H^k = (zG_\infty^{-1}z^\dagger)^{-1}$ and $h$ is a metric on $\mathcal{O}_X(1)$. 
Wang proofs that for $k \to \infty$, $H_\infty$ obeys the weak hermite Einstein equations

$$i \frac{1}{2\pi} \wedge F_{(H_\infty)} + \frac{1}{2} S(\omega) I_V = \left( \frac{deg(E)}{V_r} + \bar{s} \right) \cdot I_V,$$

where $\wedge F_{(H_\infty)}$ is the contraction of curvature form of $V$ with respect to the Kaehler form $\omega = 2i\pi Ric(h)$ on $X$, $S(\omega)$ is the scalar curvature of $X$ and $\bar{s} := \frac{1}{V} \int_X S(\omega) \frac{\omega^n}{n!}$. In particular, if $\omega$ corresponds to the Ricci flat metric, $S(\omega)$ vanishes everywhere and $H_\infty$ solves the hermite-Einstein equation.
Recall that the quintic $Q$ is given by the vanishing locus of a degree five polynomial in $\{Z_i\}$, the homogeneous coordinates on $\mathbb{P}^4$. We consider the vector bundle $V$ defined by the exact sequence

$$\mathcal{O}_Q(-4) \rightarrow \mathcal{O}_Q^\oplus 4(-1) \rightarrow V,$$

where $\mathcal{O}_Q(1)$ is the restriction of the hyperplane bundle. In particular, $V$ is given by four global section $s_i$ in $H^0(Q, \mathcal{O}_Q(3))$. We chose (for simplicity)

$$s = (Z^3_0, \ldots, Z^3_3).$$

$V$ is stable with vanishing first Chern class, hence corresponds to a valid string compactification. It is not a twist of the tangent bundle $T_Q$.

We want to find a hermitian matrix $G_\infty$ such that $T(G_\infty) = G_\infty$ where $T$ is our generalized $T$-operator. To find the global sections of $V(k)$ we consider

$$\mathcal{O}_Q(k - 4) \rightarrow \mathcal{O}_Q^\oplus 4(k - 1) \rightarrow V(k),$$
which implies the exact sequence

\[ H^0(Q, \mathcal{O}_Q(k-4)) \to H^0(Q, \mathcal{O}_Q^{\oplus 4}(k-1)) \to H^0(Q, V(k)). \]

In particular for \( k = 2 \) we find

\[ H^0(Q, V(2)) = H^0(Q, \mathcal{O}_Q^{\oplus 4}(1)) \]

that is four sets of all linear polynomials in the projective coordinates of \( \mathbb{P}^4 \). If we choose the canonical frame \( \{ \hat{e}_i \}_{i=0}^4 \) for \( \mathcal{O}_Q^{\oplus 4}(-1) \), the relation for the frame of \( V \) over the coordinate patch \( Z_0 \neq 0 \) can be expressed in inhomogenous coordinates \( w_i = Z_i/Z_0 \) as

\[ \hat{e}_0 = - \sum_{i=1}^3 w_i^3 \hat{e}_i. \]

Then, the explicit matrix \( z \) that give the embedding \( X \to Gr(3, 20) \) is given by

\[
\begin{pmatrix}
1..w_4 & 0 & 0 & -w_1^3 & -w_1^4 & -w_1^3w_2 & -w_1^3w_3 & -w_1^3w_4 \\
0 & 1..w_4 & 0 & -w_2^3 & -w_2^4 & -w_2^3w_2 & -w_2^3w_3 & -w_2^3w_4 \\
0 & 0 & 1..w_4 & -w_3^3 & -w_1w_3^3 & -w_2w_3^3 & -w_3^4 & -w_4w_3^3
\end{pmatrix}
\]

Iterating the generalized T-operator, we reach the fixed point of
the generalized T-map after 12 or 15 iterations (computation is still slow and not optimized).

One numerical result I want to mention is our computation for different rank three bundle on the Fermat quintic. Using the balanced metric $H_1$ on $V(1)$ and the Ricci flat Kaehler metric $\omega^{ij}$ we evaluate the Hermitian Yang-Mills equations. We find that

$$\omega^{ij} F_{ij} \sim 1.31 \cdot I_{3 \times 3},$$

where the theoretical value of the constant is $4/3$. That is, we indeed found a solution of hermite-Einstein metric on $V$ and get an implicit test of the Ricci flatness of $\omega^{ij}$. The error is about 11%. 

Recall that our bundle $V$ is determined by four global sections $s = (Z_0^3, \ldots, Z_3^3)$. A complex deformation corresponds simply to change of sections

$$s' = s + \delta s_V,$$

where $\delta s_V$ is an element of $H^0(Q, \mathcal{O}_Q(3))^4$. Actually, removing all trivial deformations, which are of the form

$$s_i \rightarrow s_i + \sum_j \alpha^i_j s_j$$

where $\alpha^i_j$ are constants, allows to give an explicit basis for the tangent space of the moduli space $\mathcal{M}$ at $V$. Any tangent vector $\delta s \in T_V \mathcal{M}$ can be expressed as

$$\delta s_V = \sum_{ij} a_{ij} m_j e_i$$

where $a_{ij}$ are constants, $\{e_i\}$ form the canonical basis of $\mathbb{C}^4$ and $\{m_j\}_{j=1}^{31}$ is the set of all degree three monomials in $\{Z_i\}$ modulo $Z_0^3, \ldots, Z_3^3$. 

A variation of $\delta s_V$ leads to $\delta z_V$, a variation of the embedding of $X \to Gr(3, 20)$. In particular we obtain a different fixed point of the $T$-operator

$$G' = G + \delta G_V.$$  

Using the condition $T(G_V) = G_V$ allows to solve for $\delta_V G$.

Recall that the hermite Einstein metric on $V$ was given by

$$H = (zG^{-1}z^\dagger)^{-1}h^{-k},$$

hence we can explicitly solve for $\delta_V H$. To compute the Kaehler metric one has to find $\delta_V A$ in the physical gauge where the connection one forms are antihermitian. This can be done solving the equation

$$H = C^\dagger C.$$

The connection one forms are given by

$$A_{\bar{i}} = C(\bar{\partial}_i C^{-1}).$$

We solve for $\delta_V A$ and compute the normalization matrix

$$g_{p\bar{q}} = \int_X d^6tr(\delta_p A_i, \delta_{\bar{q}} A_{\bar{j}})g^{ij}.$$