Towards a complete 4-dim Lagrangian form Heterotic string theory

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Motivation

We are interested in compactifications of het. string theory to $N = 1$, 4-dim effective theories. In particular, we want to write an effective Lagrangian.

To find such 4-dim supersymmetric theories, one can the following:

1. Choose a space time $M^{10} = X \times M^4$, M^4 is 4-dim Mink. Take X to be CY-space (complex Kaehler manifold with trivial canonical bundle)
   In particular, we want to find a Ricci-flat metric $g_{i\bar{j}}$ on X. If we fix a Kaehler class $H = [g_{i\bar{j}}]$, a theorem of Yau ensures the existence of a unique such metric.
2. Solve hermitian Yang-Mills equation on X, that is, find a gauge potential A_i, such that

$$F_{ij}(A) = 0,\ F_{\bar{i}\bar{j}}(A) = 0,\ g^{i\bar{j}} F_{i\bar{j}}(A) = 0.$$

   Such A_i can be interpreted as connection of holomorphic vector bundle V.

Theorems of Donaldson-Uhlenbeck-Yau show, that if V is a holomorphic bundle, which is stable with respect to H, then there exists a unique such solution A_i.

3. Find the field content of the 4-dim theory. It is determined by the zero modes ψ^α of the Dirac operator on X. This amounts to find harmonic differential forms ψ valued in V, $i.e.$ solutions of

$$0 = (\bar{\partial} + \bar{A})\psi = (\bar{\partial} + \bar{A})^* \psi,$$

where $*$ denotes the adjoint operator. These modes can be described algebraically as the sheaf cohomology of V.

4. Find 4-dim superpotential, that is, the wedge products of differential forms ψ_α. This also can be done algebraically, e.g. rank three case, one needs to evaluate the map

$$H^1(X, V) \otimes H^1(X, V) \otimes H^1(X, V) \to H^3(X, \wedge^3 V) = \mathbb{C}.$$

5. Find the Kaehler potential for eff. 4-dim theory, or at least, the Kaehler metric which gives normalization for the 4-dim particles.

For example, let consider the normalization for the massless 4-dim fields which correspond to the moduli of the hermitian Yang-Mills gauge potential. Consider a deformation δA of a solution A_0 to the HYM equation

$$A' = A_0 + \delta A.$$

That is, A' solves the HYM equations as well. These deformations correspond to harmonic representatives of the cohomology classes of $H^1(X, V \otimes V^*)$ and give rise in 4-dim. to complex scalar fields ϕ. After dimensional reduction of the 10-dim Yang-Mills action, on obtains

$$S_{kin}(\phi) = \sum \int_X d^6 tr(\delta_p A_i, \delta_{\bar{q}} A_{\bar{j}}) g^{i\bar{j}} \int_{M_4} d^4 x \partial_\mu \phi_p \partial^\mu \phi^*_{\bar{q}}$$

where $p, \bar{q}$ run over a basis of $H^1(X, V \otimes V^*)$ and denote the projections to the $(1,0)$-form and $(0,1)$-form part of δA respectively. $g_{i\bar{j}}$ is the Ricci flat metric on X.

In this talk I present our method on how to obtain numerical solutions to hermite Yang-Mills equation for stable bundles which can be applied on any compact Kaehler manifold.

I show how to apply this method to a stable rank three bundle V on a quintic.

In addition, I present a method on how to compute the Kaehler metric

$$g_{p\bar{q}} = \int_X d^6 tr(\delta_p A_i, \delta_{\bar{q}} A_{\bar{j}}) g^{i\bar{j}}$$

numerically. There are generalizations of this method that can be applied to obtain the normalization of all 4-dim fields.

Review Ricci flat metrics

A a warm up, lets review the numerical computation of Ricci flat metrics on quintics, as described yesterday by Robert Karp. The basic idea is to consider embeddings of a quintic into a complex projective space of high dimension N

$$X \subset \mathbb{P}^N.$$

Projective spaces have a set a large set of Kaehler metrics, described by the Kaehler potentials

$$K_h = log\left(\sum_{i\bar{j}} h^{i\bar{j}} Z_i \bar{Z}_j\right),\ i\bar{j} = 0, \ldots N.$$

Note that the dimension of the family of Kaehler potential grows like N^2, hence for large N there is a large such family. The idea is, as suggested by Yau, Tian, Luo and Donaldson, that the pull-back of the "correct" Kaehler metric given by K_{h_c} is a good approximation to the Ricci flat metric and agrees with it in the limit $N \rightarrow \infty$.

More formally, the embedding is given the global sections s_α of the k tensor power of the hyperplane bundle $\mathcal{O}_X(1)$ using the evaluation map

$$x \longrightarrow (s_1(x), \ldots, s_N(x)) \in \mathbb{C}^N.$$

Donaldson showed recently that the "correct" Kaehler metric on $\mathbb{P}^N$ is given by the fixed point $T(h) = h$ of the T-operator

$$T(h)_{\alpha\bar{\beta}} \equiv \frac{N+1}{vol(X)} \int_X dV \; \frac{s_\alpha \bar{s}_{\bar{\beta}}}{\sum h^{\alpha\bar{\beta}} s_{\alpha\bar{\beta}}}.$$

and that $T^n(h)$ converges for $n \rightarrow \infty$.

Solution to hermitian Yang-Mills equation

To apply this strategy to solve for the hermitian Yang-Mills connection A_i, recall that on a complex vector bundle V, there is a correspondence between a hermitian metric $(\,,\,) : \bar{V} \otimes V \longrightarrow \mathbb{C}^{\infty}(X)$, which in a frame $\{e_b\}$ gives $H_{a\bar{b}} = (e_a, e_b)$ and its corresponding metric connection $A_i = H^{-1}\partial_i H$.

Instead of solving for the hermitian Yang-Mills connection A_i, we solve for the hermite-Einstein metric H in the gauge $\bar{A} = 0$

Hence we want to find a large family of metrics on the holomorphic vector bundle V (with fixed rank r). We consider the bundle $V \otimes \mathcal{O}_X(k)$ with M global sections such that the map $X \longrightarrow G(r, M)$ defines an embedding into the Grassmanian $G(r, M)$.

More precisely, after choosing a local frame for V, a basis of sections determines a $M \times r$ matrix $\{z^a_\alpha\}^{a=1,\ldots,r}_{\alpha=1,\ldots,M}$. It is defined up to a $GL(M)$ change of basis and up to a $GL(r)$ change of frames, hence gives for each $x \in X$ we obtain a point in the Grassmanian $G(r, M)$.

Now we get a set of natural metrics on $V(k)$

$$H = (zG^{-1}z^{\dagger})^{-1}.$$

parameterized by $M \times M$ matrix G. Again, one wants to find a natural metric of this form which is a good approximation to hermite-Einstein metric and agrees with it in the large volume limit.

To find this metric, we define a generalized T-operator (DKLR)

$$T(G) = \frac{N}{Vr} \int_X z(z^* G^{-1} z)^{-1} z^* dV.$$

and conjecture that for any stable bundle V it convergence to a fixed point G_∞. We test this conjecture for $T\mathbb{P}^2$, a rank three bundle on $\mathbb{P}^2$ and rank three bundles on quintics. Such fixed point G_∞ leads to a natural metric on V for any k

$$H_k = H^{(k)} \otimes h^{-k},$$

where $H^k = (zG_\infty^{-1}z^{\dagger})^{-1}$ and h is a metric on $\mathcal{O}_X(1)$.

Wang proofs that for $k \to \infty$, H_∞ obeys the weak hermite Einstein equations

$$\frac{i}{2\pi} \bigwedge F_{(H_\infty)} + \frac{1}{2} S(\omega) I_V = \left(\frac{deg(E)}{Vr} + \frac{\bar{s}}{2} \right) \cdot I_V,$$

where $\bigwedge F_{(H_\infty)}$ is the contraction of curvature form of V with respect to the Kaehler form $\omega = 2i\pi Ric(h)$ on X, $S(\omega)$ is the scalar curvature of X and $\bar{s} := \frac{1}{V} \int_X S(\omega) \frac{\omega^n}{n!}$. In particular, if ω corresponds to the Ricci flat metric, $S(\omega)$ vanishes everywhere and H_∞ solves the hermite-Einstein equation.

Example: Hermite Einstein metric on V

Recall that the quintic Q is given by the vanishing locus of a degree five polynomial in $\{Z_i\}$, the homogenous coordinates on $\mathbb{P}^4$. We consider the vector bundle V defined by the exact sequence

$$\mathcal{O}_Q(-4) \rightarrow \mathcal{O}_Q^{\oplus 4}(-1) \rightarrow V,$$

where $\mathcal{O}_Q(1)$ is the restriction of the hyperplane bundle. In particular, V is given by four global section s_i in $H^0(Q, \mathcal{O}_Q(3))$. We chose (for simplicity)

$$s = (Z_0^3, \ldots, Z_3^3).$$

V is stable with vanishing first Chern class, hence corresponds to a valid string compactification. It is not a twist of the tangent bundle T_Q.

We want to find a hermitian matrix G_∞ such that $T(G_\infty) = G_\infty$ where T is our generalized T-operator. To find the global sections of $V(k)$ we consider

$$\mathcal{O}_Q(k-4) \rightarrow \mathcal{O}_Q^{\oplus 4}(k-1) \rightarrow V(k),$$

which implies the exact sequence

$$H^0(Q, \mathcal{O}_Q(k-4)) \to H^0(Q, \mathcal{O}_Q^{\oplus 4}(k-1)) \to H^0(Q, V(k)).$$

In particular for $k = 2$ we find

$$H^0(Q, V(2)) = H^0(Q, \mathcal{O}_Q^{\oplus 4}(1))$$

that is four sets of all linear polynomials in the projective coordinates of $\mathbb{P}^4$. If we choose the canonical frame $\{\hat{e}_i\}_{i=0}^4$ for $\mathcal{O}_Q^{\oplus 4}(-1)$, the relation for the frame of V over the coordinate patch $Z_0 \neq 0$ can be expressed in inhomogenous coordinates $w_i = Z_i/Z_0$ as

$$\hat{e}_0 = -\sum_{i=1}^{3} w_i^3 \hat{e}_i.$$

Then, the explicit matrix z that give the embedding $X \to Gr(3, 20)$ is given by

$$\begin{pmatrix} 1..w_4 & 0 & 0 & -w_1^3 & -w_1^4 & -w_1^3 w_2 & -w_1^3 w_3 & -w_1^3 w_4 \\ 0 & 1..w_4 & 0 & -w_2^3 & -w_1 w_2^3 & -w_2^4 & -w_3 w_2^3 & -w_4 w_2^3 \\ 0 & 0 & 1..w_4 & -w_3^3 & -w_1 w_3^3 & -w_2 w_3^3 & -w_3^4 & -w_4 w_3^3 \end{pmatrix}$$

Iterating the generalized T-operator, we reach the fixed point of the generalized T-map after 12 or 15 iterations (computation is still slow and not optimized).

One numerical result I want to mention is our computation for different rank three bundle on the Fermat quintic. Using the balanced metric H_1 on $V(1)$ and the Ricci flat Kaehler metric $\omega^{i\bar{j}}$ we evaluate the Hermitian Yang-Mills equations. We find that

$$\omega^{i\bar{j}} F_{i\bar{j}} \sim 1.31 \cdot \mathbf{I}_{3\times 3},$$

where the theoretical value of the constant is $4/3$. That is, we indeed found a solution of hermite-Einstein metric on V and get an implicit test of the Ricci flatness of $\omega^{i\bar{j}}$. The error is about 11%.

Moduli space of V

Recall that our bundle V is determined by four global sections $s = (Z_0^3, \ldots, Z_3^3)$. A complex deformation corresponds simply to change of sections

$$s' = s + \delta s_V,$$

where δs_V is an element of $H^0(Q, \mathcal{O}_Q(3))^4$. Actually, removing all trivial deformations, which are of the form

$$s_i \longrightarrow s_i + \sum_j \alpha_i^j s_j$$

where α_i^j are constants, allows to give an explicit basis for the tangent space of the moduli space $\mathcal{M}$ at V. Any tangent vector $\delta s \in T_V \mathcal{M}$ can be expressed as

$$\delta s_V = \sum_{ij} a_{ij} m_j e_i$$

where a_{ij} are constants, $\{e_i\}$ form the canonical basis of $\mathbb{C}^4$ and $\{m_j\}_{j=1}^{31}$ is the set of all degree three monomials in $\{Z_i\}$ modulo $Z_0^3, \ldots, Z_3^3$.

A variation of δs_V leads to δz_V, a variation of the embedding of $X \to Gr(3,20)$. In particular we obtain a different fixed point of the T-operator

$$G' = G + \delta G_V.$$

Using the condition $T(G_V) = G_V$ allows to solve for $\delta_V G$.

Recall that the hermite Einstein metric on V was given by

$$H = (zG^{-1}z^{\dagger})^{-1}h^{-k},$$

hence we can explicitly solve for $\delta_V H$. To compute the Kaehler metric one has to find $\delta_V A$ in the physical gauge where the connection one forms are antihermitian. This can be done solving the equation

$$H = C^{\dagger}C.$$

The connection one forms are given by

$$A_{\bar{i}} = C(\bar{\partial}_{\bar{i}}C^{-1}).$$

We solve for $\delta_V A$ and compute the normalization matrix

$$g_{p\bar{q}} = \int_X d^6 tr(\delta_p A_i, \delta_{\bar{q}} A_{\bar{j}}) g^{i\bar{j}}.$$