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Non-Abelian Duality from Vortex Moduli

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1. Introduction to Non-Abelian Duality
2. Brief Review on Non-Abelian Monopole and Vortex
3. Non-Abelian Duality from Monopole-Vortex Complex
4. Discussions

Collaboration with

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1 Introduction to Non-Abelian Duality

- Electric-Magnetic Duality is a Powerful Tool for Non-Perturbative Analysis

$$\mathbf{E} \Longleftrightarrow \mathbf{B}$$

$$e \Longleftrightarrow g = 1/e$$

Actually, the Duality leads to Exact Solutions for $\mathcal{N} = 2$ Super-Yang-Mills Theory.

$\Longrightarrow$ (Abelian) Monopole Condensation $\sim$ Dual Meissner Effect

However, (Simple) Abelian Effective Theory Has Different Dynamical Properties from QCD.

$\Longleftarrow$ E.g., Richer Hadron Spectrum, $\cdots$

$\Diamond$ Need of Non-Abelian Effective Description

and

Non-Abelian Duality

An Example of Non-Abelian Effective Description

$\Longrightarrow$ So-Called r-Vacua with $SU(r) \times U(1)^{N_c - r}$ Sym.

- An Example of Non-Abelian Duality:

Goddard-Nuyts-Olive-Weinberg (GNOW) Duality

For System with the Breaking Pattern, $G \Longrightarrow H$ $(H$: Non-Abelian) ,

GNOW Duality:

$$H \Longleftrightarrow H^*$$
$$\alpha \Longleftrightarrow \alpha^* = \frac{\alpha}{\alpha \cdot \alpha}$$

★ H^* : DUAL Group Generated by DUAL Root α^*

Example :

$SU(N)$	$\Leftrightarrow$	$SU(N)/Z_N$
$SO(2N)$	$\Leftrightarrow$	$SO(2N)$
$SO(2N+1)$	$\Leftrightarrow$	$USp(2N)$

Note : $U(N)$ is Self-Dual.

- Evidence for GNOW Duality : Non-Abelian Monopole
  - Topological Argument :

$\pi_2(G/H)$ is Non-Trivial $\Longrightarrow$ Regular Solitonic Monopoles.

Asymptotic Behavior of Solution at $r \sim \infty$ ($U \in G$, $T_i \in$ C.S.A. of H)

$$\phi \sim U\langle\phi\rangle U^{-1}, \quad F_{ij} \sim \epsilon_{ijk}\frac{x^k}{r^3}(\beta \cdot T) .$$

◊ Generalized Dirac Quantization Condition :

$2\alpha \cdot \beta \in \mathbb{Z}$ for Roots α of H.

β gives a Weight Vector of H^* $\Longrightarrow$ Monopole Forms a Multiplet of H^*

We *Define* the Dual Transformation among these Non-Abelian Monopoles.

In Fact, This is NOT an Easy Task as You See...

2 Brief Review on Non-Abelian Monopole and Vortex

- (Semi-)Classical Solution for Non-Abelian Monopole

Simple Example : $SU(3)$ Yang-Mills Theory with Ajoint Higgs Φ.

$$SU(3) \overset{\langle\Phi\rangle}{\Longrightarrow} \frac{SU(2)\times U(1)}{Z_2} \quad \text{by} \quad \langle\Phi\rangle = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -2v \end{pmatrix}$$

In This Case, $\pi_2(G/H) \sim \pi_1\left(\frac{SU(2)\times U(1)}{Z_2}\right) = \mathbb{Z}$.

Regular BPS Solitonic Solution :

$$\begin{aligned} \Phi(x) &= \begin{pmatrix} -\frac{1}{2}v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -\frac{1}{2}v \end{pmatrix} + 3v\,\vec{S}\cdot\hat{r}\,\phi(r) \\ \vec{A}(x) &= \vec{S}\times\hat{r}\,A(r), \end{aligned}$$

where $\phi(r),\ A(r)$ are BPS-'t Hooft's Profile Function.

- $\vec{S}$ is a Minimal Embedded $SU(2)$ Algebra in (1, 3) and (2, 3) Subspaces.

$\Diamond$ Two Degenerate Solutions $\Rightarrow$ Doublet of Dual $SU(2)$?

In Fact, These Two are Continuously Connected by Unbroken $SU(2)$ Transformation.

Multiplicity of the Monopoles are 1 or 2 or ∞ ?

In Order to Answer the Question, Need to Understand the Tranformation Properties.

However, Some Difficulties are Well-Known in Semi-Classical Analysis for the Solutions

- Non-Normalizable Zero-Modes Appear due to Unbroken $SU(2)$.

- There exists Topological Obstacle to Definition of Charge of the $SU(2)$.

Standard Quantization Procedures Break Down due to the Difficulties.

How can We Overcome These Situations ?

◊ Our Idea : Consider the System with Hierarchical Symmetry Breaking

$$G \xrightarrow{v_1} H \xrightarrow{v_2} \emptyset, \qquad v_1 \gg v_2.$$

In this System with $\pi_2(G/H) \neq 0$, Everything Goes Better.

1. At High Energy ($\geq v_1$), $G \longrightarrow H$ Breaking Produces Non-Abelian Monopoles.

2. At Low Energy ($\sim v_2$), Breaking of H Produces Non-Abelian Vortices.

Non-Abelian Monopoles are Confined by Non-Abelian Vortex !

Some Comments :

- Low Energy H-Theory is in Higgs Phase $\Rightarrow$ DUAL Theory is in Confining Phase.
  (Cf. H^* is in Higgs Phase $\Rightarrow$ NO Multiplet Structure)

- Non-Normalizable Zero-Modes Should be Normalizable in $M\bar{M}$-System.

- Light Higgs in the Fundamental Rep. is Needed for Breaking of H.
  $\Longrightarrow$ Massless "Flavor" is Crucial for Non-Abelian Duality (See Later)

★ Our Model : Softly Broken $\mathcal{N} = 2$ Supersymmetric QCD

$\mathcal{N} = 2\; SU(N+1)$ Gauge Theory with N_f Fundamental Hypermultiplets and

$$\mathcal{N} = 1 \text{ Breaking Term} \;:\quad \Delta W = \int d^2\theta\, \mu\, \mathrm{Tr}\, \Phi^2 .$$

Set Bare Mass Parameter for Hypermultiplets ${}^{\forall} m_i = m \quad (i = 1, 2, \cdots, N_f)$.

- The r-Vacuum with $r = N$:

$$\Phi = -\frac{1}{\sqrt{2}} \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & \ddots & \vdots & \vdots \\ 0 & \dots & m & 0 \\ 0 & \dots & 0 & -N\, m \end{pmatrix},$$

$$Q = \tilde{Q}^{\dagger} = \begin{pmatrix} d & 0 & 0 & 0 & \dots \\ 0 & \ddots & 0 & \vdots & \dots \\ 0 & 0 & d & 0 & \dots \\ 0 & \dots & 0 & 0 & \dots \end{pmatrix}, \quad d = \sqrt{(N+1)\, \mu\, m}.$$

◊ For $m \gg \mu$,

- Φ Breaks $SU(N+1) \Rightarrow \frac{SU(N)\times U(1)}{Z_N}$ at $v_1 \sim m$
- Q Breaks $\frac{SU(N)\times U(1)}{Z_N}$ Completely at $v_2 \sim d$.

★ However, Diagonal $SU(N)_{C+F} \subset SU(N) \times SU(N_f)$ Sym. is Preserved.

- Low Energy Effective Theory at $(v_2 \lesssim) E \ll v_1$

$\mathcal{N} = 2 \ SU(N) \times U(1)$ Gauge Theory with N_f "Quarks" and "FI-Term".

- Bosonic Part is Much Similar to the Lagrangian in Eto's Talk.
  (Except for Difference of Couplings Between $SU(N)$ and $U(1)$)
- $\pi_1(SU(N) \times U(1)/Z_N) = \mathbb{Z}$

$\Longrightarrow$ BPS Non-Abelian Vortex Solution.

◊ Moduli Matrix Formalism for Non-Abelian Vortex

Equation of Motion : (q is the First N Squarks in N_f-Flavors)

$$(\mathcal{D}_1 + i\mathcal{D}_2)\, q = 0,$$

$$F_{12}^{(0)} + \frac{e^2}{2}\left(c\,\mathbf{1}_N - q\,q^\dagger\right) = 0, \qquad F_{12}^{(a)} + \frac{g_N^2}{2}\, q_i^\dagger\, t^a\, q_i = 0.$$

Solutions ($z = x_1 + ix_2$):

$$q = S^{-1}(z,\bar{z})\, H_0(z), \quad A_1 + i\, A_2 = -2\, i\, S^{-1}\, \bar{\partial}_z S(z,\bar{z}).$$

- $S(z,\bar{z})$ Satisfies a Nonlinear “Master Equation”.
- $H_0(z)$ is Moduli Matrix which Encodes All Moduli Parameters,

  up to the V-Transformation : $H_0(z) \to V(z) H_0(z)$ (V is any Hol. Matrix).

Examples of Moduli Spaces

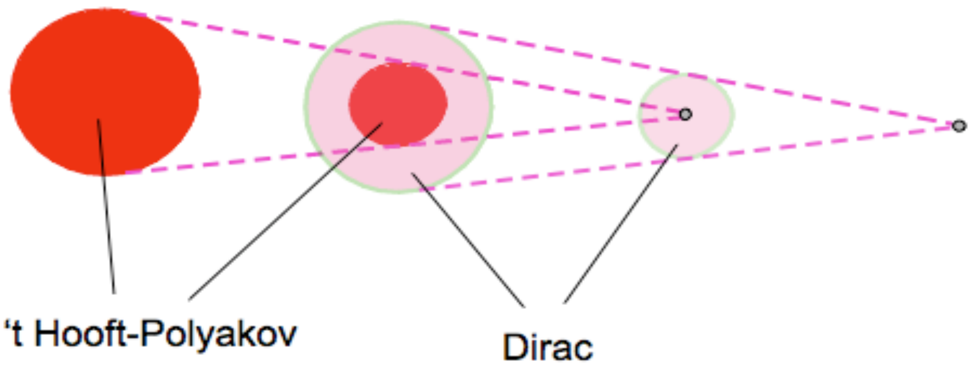
1. 1-Vortex for $SU(N) \times U(1)$ Theory : $\mathcal{M} = \mathbb{C}P^{N-1}$.
2. Composite 2-Vortex in $SU(2) \times U(1)$ Theory : $\mathcal{M}_{k=2} = W\mathbb{C}P^2_{(2,1,1)}$.

3 Non-Abelian Duality from Monopole-Vortex Complex

◇ Monopole-Vortex Complex from Topological Argument

Exact Homotopy Sequence :

$$\ldots \to \pi_2(G) \to \pi_2(G/H) \to \pi_1(H) \to \pi_1(G) \to \ldots$$



In Our Case, $\pi_2(G) = 0$ and $\pi_1(G) = 0$.

$$\pi_2\left(\frac{SU(N+1)}{U(N)}\right) = \pi_2(\mathbf{C}P^N) \sim \pi_1(U(N)) = \mathbb{Z}$$

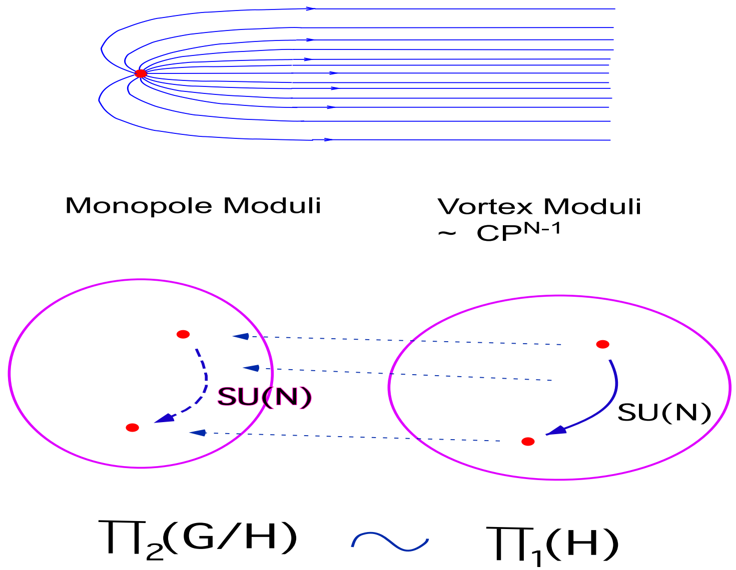
High-Energy Monopole $\Longleftrightarrow$ Low-Energy Vortex

◇ Dual Transformations among Monopoles

A Vortex Solution Breaks Color-Flavor Diag. Sym.

$$SU(N)_{C+F} \longrightarrow SU(N-1) \times U(1)$$

- Moduli Space for 1-Vortex : $\mathcal{M} = SU(N)/U(N-1) = \mathbb{C}P^{N-1}$.



★ We can Show the Moduli Parameters Transforms N-Rep. under $SU(N)_{C+F}$.

$\Longrightarrow$ High-Energy Non-Abelian Monopoles Form an N-Rep Multiplet.

- Simplest Example for $SU(2)\times U(1)$ Theory

Moduli Matrix up to V-Transformation

$$H_0^{(1,0)} \simeq \begin{pmatrix} z-z_0 & 0 \\ -b_0 & 1 \end{pmatrix}, \quad H_0^{(0,1)} \simeq \begin{pmatrix} 1 & -a_0 \\ 0 & z-z_0 \end{pmatrix}.$$

- a_0 and b_0 are Orientational Moduli and Correspond to Two Patches of $\mathbb{C}P^1$.
- Under $SU(2)_{C+F}$ Transformation :

$$H_0 \to V\, H_0\, U^\dagger, \qquad U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad (|\alpha|^2+|\beta|^2=1),$$

Moduli Parameter a_0 Transforms as

$$a_0 \longrightarrow \frac{\alpha\, a_0+\beta}{\alpha^*-\beta^*\, a_0}.$$

★ This is Nothing But the Transformation of Doublet.

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \to \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad a_0 \equiv \frac{a_1}{a_2}.$$

◇ This Derivation Does NOT Depend on Semi-Classical Analysis of Monopole

★ Another Non-Trivial Example : $SO(2N+1) \longrightarrow U(N) \longrightarrow \emptyset$

- Simplest Case for $SO(5) \longrightarrow U(2) \longrightarrow \emptyset$.

  Essential Differences : $\pi_1(SO(5)) = Z_2$

  - Minimal Monopole is Dirac-Type and Minimal Vortex is Truly Stable.

(1). Vortex Side : We have Investigated Moduli Space of Composite 2-Vortex (Eto's Talk)

$$\mathcal{M}_{k=2} = W\mathbb{C}P^2_{(2,1,1)} \simeq \mathbb{C}P^2/Z_2.$$

- Bulk of $W\mathbb{C}P^2$: **Triplet** under $SU(2)_{C+F}$.
- Conical Singularity : **Singlet**.

(2). Monopole Side : Regular Solution with One Parameter Not Related to Sym. (E. Weinberg)

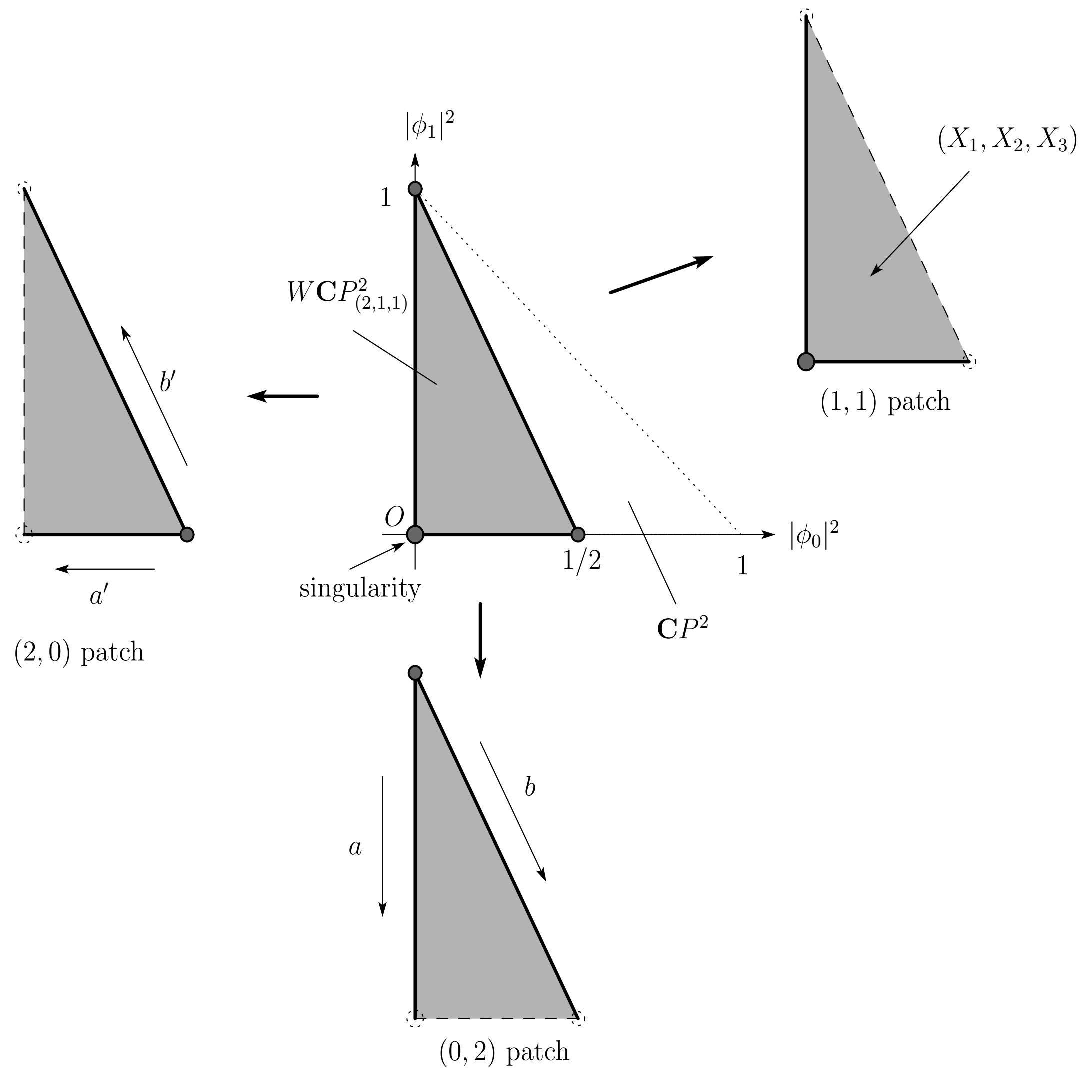
Fortunately, Moduli Space and Metric is KNOWN,

$$\mathcal{M}_{\text{mono}} = \mathbb{C}^2/Z_2 \simeq H_0^{(1,1)} \quad : \quad \text{A Patch of } W\mathbb{C}P^2$$

- A "Compactification" of $\mathcal{M}_{\text{mono}}$ Gives $W\mathbb{C}P^2$.

★ Monopoles Transform : $\mathbf{3} \oplus \mathbf{1} (= \mathbf{2} \otimes \mathbf{2})$.

- Moduli Space of Composite 2-Vortex in $\boldsymbol{SU(2)} \times \boldsymbol{U(1)}$ Theory (Eto's Talk)



◊ Dual Symmetry as Color-Flavor Diagonal Symmetry

- Color-Flavor Diagonal Sym. $SU(N)_{C+F}$ is EXACT Symmetry of the Theory.

  $\Longrightarrow$ Energy of Whole Monopole-Vortex System is Invariant.

- In High Energy Theory $(v_2 \longrightarrow 0)$, This Sym. Acts as ONLY Color Part of $SU(N)_{C+F}$.

  $\Longrightarrow$ In Full Theory, This Sym. Becomes Non-Local Sym. Involving Flavor !

★ Dual Transformation as Non-Local Transformation by $SU(N)_{C+F}$

Note : Flavor Dependence of Dual Sym. is Well-Known in Seiberg-Duality.

◊ Quantum Aspects of Non-Abelian Duality

In Full-Quantum Theory, This Dual Sym. $SU(N)_{C+F}$ Has Trouble.

- According to Famous Seiberg-Witten Results,

  Strong Coupling Dynamics Breaks $SU(N)$ to ABELIAN $U(1)^{N-1}$.

In Order to Resolve this, $N_f \geq 2N$ Massless Flavors are Crucial.

Note : In such a Case, Low-Eenrgy Theory Becomes Infra-Red Free.

4 Summary and Discussion

Summary

- We Have Discussed the (Non-Abelian) Dual Transformation among Non-Abelian Monopoles through Studying Non-Abelian Monopole-Vortex Complex. Using the Moduli Matrix Formalism for Non-Abelian Vortex, We Have Determined Transformation Properties of Non-Abelian Monopoles under $SU(N)_{C+F}$ Symmetry as a Dual Symmetry.

Discussion and Future Problems

- Deeper Understanding of Relation Between Dual Sym. and $SU(N)_{C+F}$.
- Analysis of Moduli Space of Semi-Local Non-Abelian Vortex ($N_f > N$)
- Analysis of the Case $SO(2N+3) \to SO(2N+1) \to \emptyset$.
- Seiberg's Dual Quarks as Non-Abelian Monopoles ?