B and D Semileptonic Decays on the Lattice

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Thanks, Richard Hill, Ruth Van de Water
Exclusive semileptonic decays on the lattice

provide good determinations of CKM matrix elements.

Tune $V_{ub}$ to get correct $B \rightarrow \pi l \nu$.

$$\Gamma \left( \bar{B} \rightarrow X u \ell \bar{\nu}_\ell \right) = G_2^2 |V_{ub}|^2 m_5^b \left( 1 - \frac{4}{3} \alpha_s \right) $$

$$\left( 1 - \frac{2}{3} \alpha_s \right)$$

Decays: short distance (calculable)

Hadrondization: long distance (nonperturbative) - but at leading order, long and short distances are cleanly separated and probability to hadronize is unity.

The basic theoretical tools are more than a decade old.

$$X = \{ \pi, K, \ldots \}$$
Outline

• Overview of Fermilab/MILC semileptonic program

• Constrained curve fitting

• Constrained curve fitting and the shape of semileptonic form factors.
Current Fermilab/MILC semileptonic projects and writeups.

<table>
<thead>
<tr>
<th>Process</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D \rightarrow {\pi, K} l \nu)</td>
<td>Phys. Rev. Lett. 94:011601, 2005</td>
</tr>
<tr>
<td>(B \rightarrow D l \nu)</td>
<td>Okamoto, Lattice 05</td>
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<tr>
<td>(B \rightarrow D^* l \nu)</td>
<td>Laiho, Lattice 06</td>
</tr>
<tr>
<td>(B \rightarrow \pi l \nu)</td>
<td>Masataka Okamoto, Lattice 05</td>
</tr>
<tr>
<td></td>
<td>Van de Water, Lattice 06</td>
</tr>
<tr>
<td>(B \rightarrow K l^+ l)</td>
<td>Jain, Lattice 06</td>
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</tbody>
</table>
Long-term plan is to analyze all of these on the MILC lattices with $a=0.15$, 0.125, and 0.09 fm.

<table>
<thead>
<tr>
<th>$a$(fm)</th>
<th>L</th>
<th>$m_l$</th>
<th>$m_s$</th>
<th>$m_\pi$(MeV)</th>
<th>approx. # configs.</th>
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<td>0.0484</td>
<td>0.0484</td>
<td>700</td>
<td>600</td>
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</table>
\[ B \rightarrow D^{(*)} l \nu \]

**B->D* RUN PARAMETERS:**

**Fine Lattices:**
- Coming soon

**Coarse Lattices:**
- \( t_{\text{sink}} - t_{\text{source}} = 12 \)
- \( t_{\text{source}} = 0, 16, 32, 48 \)
- smeared heavy clover daughter quark
- local heavy clover parent quark
- local staggered spectator quark
- full QCD only
- heavy kappas = (0.074, 0.086, 0.093, 0.119, 0.114, 0.122)

**Medium-Coarse Lattices:**
- \( t_{\text{sink}} = 10? \)
- \( t_{\text{source}} = 0, 24 \) (also 12, 36?)
- full QCD only

**AVAILABLE 3pt DATA:**

**Coarse Lattices:**
- 0.02/0.05 ensemble -- \( t_{\text{source}} = 0, 16, 32, 48 \)
- 0.01/0.05 ensemble -- \( t_{\text{source}} = 0, 16, 32, 48 \)
- 0.007/0.05 ensemble -- \( t_{\text{source}} = 0, 16, 32, 48 \)

**Medium-Coarse Lattices:**
- 0.0194/0.0484 ensemble (In progress)
- 0.0290/0.0484 ensemble (In progress)

**Staggered chiral PT,**
Laiho and Van de Water,
Phys. Rev. **D73:**054501, 2006

Currently working on a run at 
\( a=0.15 \) fm, to obtain an estimate of discretization errors, before moving on to \( a=0.09 \) fm.
B→π RUN PARAMETERS:

Coarse Lattices:

t_sink = 12
    t_source = 0, 32 (also 16, 48?)
    local pion
    smeared B
    full QCD only

Medium-Coarse Lattices:

t_sink = 10?
    t_source = 0, 24 (also 12, 36?)
    local pion
    smeared B
    full QCD only

AVAILABLE 3pt DATA:

Coarse Lattices:

0.02/0.05 ensemble -- t_source = 0, 16, 32, 48
0.01/0.05 ensemble -- t_source = 0, 32
0.007/0.05 ensemble -- t_source = 0, 32

Medium-Coarse Lattices:

0.0194/0.0484 ensemble -- t_source = 0, 32 @ t_sink = 12; t_source = 0 @ t_sink = 8, 10
0.0290/0.0484 ensemble -- t_source = 0 @ t_sink = 8, 10, 12

Currently working on a run at $a=0.15$ fm, to obtain an estimate of discretization errors, before moving on to $a=0.09$ fm; studying optimal ways of performing unitarity-based fits (see later).
\[
\langle B(p)\big|\bar{b}\gamma^\mu s\big|K(k)\rangle = f_+(p^\mu + k^\mu - \frac{m_B^2 - m_K^2}{q^2}q^\mu) + f_0\frac{m_B^2 - m_K^2}{q^2}q^\mu
\]

\[
\langle B(p)\big|\bar{b}\sigma^{\mu\nu}q_\nu s\big|K(k)\rangle = i \frac{f_T}{m_B + m_K} \{ (p + k)^\mu q^2 - q^\mu(m_B^2 - m_K^2) \}
\]

New operators and form factors in the Standard Model.

Standard Model effects are small.
(Proceed through penguins.)
Possibility of seeing Beyond-the-Standard-Model effects?

Penguin diagrams contributing to \( B \to K \ell^+ \ell^- \)
Very preliminary.

Form Factors

- $f_0(q^2)$
- $f_+(q^2)$
- $f_T(q^2)$

$q^2 (\text{GeV}^2)$

Conclusion and Outlook

Though our analysis is still being developed, the above work demonstrates a potential framework for making a lattice prediction of the form factors leading to $B \rightarrow K \ell^+ \ell^-$. The first step in improving our analysis will be to study the systematic errors introduced by fitting. We plan to study in detail the robustness of our procedure, as well as to explore alternate methods such as fitting to the ratio of correlators as has been done elsewhere in our collaboration. It is hoped that by using multiple time-sources, as well as combining smeared and unsmeared data, we can reduce the statistical noise in our data, allowing us to better constrain excited states and improve the quality of our fits. This analysis will also need to be repeated on other MILC ensembles so that we can study the lattices spacing and light-quark mass dependence of our results. Additionally, we will need to include the perturbative corrections to the relevant matrix elements, and perform a chiral extrapolation down to the physical light quark mass.
$B \rightarrow \pi l \nu$, finite range of $q^2$

Uncertainties in lattice and experiment both highly $q^2$ dependent. Harder and more important to understand shape.

Proposals to address:

*) Moving NRQCD (Davies, Lepage, et al.)

*) Calculate in charm region, extrapolate to bottom (Abada et al.)

*) Gibbons: global simultaneous fit of all experimental and lattice data.

*) Unitarity and analyticity (Lellouch, Fukunaga-Onogi, Arnesen et al., ...)

**Lattice** data extend over only a fraction of the $q^2$ range on the physical $B \rightarrow \pi l \nu$ decay.

With standard methods, discretization errors go like $O(ap)^2$, signal goes like $\exp(-E_{\pi}t)$.
Always have to make some assumptions about shape to obtain any fit.

If any crazy shape were allowed, we could never fit anything.

Constrained curve fitting allows the possibility of an infinite number of free parameters in fits, while formalizing assumptions about their values.
Constrained curve fitting

\[ \chi^2 \rightarrow \chi^2_{\text{aug}} \equiv \chi^2 + \chi^2_{\text{prior}} \]

Add an infinite number of parameters to the fit function, but constrain them to their plausible ranges.

\[ \chi^2(A_n, E_n) \equiv \sum_{t,t'} \Delta G(t) \sigma_{t,t'}^{-2} \Delta G(t') \]

Use “augmented” chi squared:

\[ \chi^2_{\text{prior}} \equiv \sum_n \frac{(A_n - \tilde{A}_n)^2}{\tilde{\sigma}_A^2} + \sum_n \frac{(E_n - \tilde{E}_n)^2}{\tilde{\sigma}_E^2} \]

Bayes formula:

\[ P(\rho|\overline{G}) = \frac{P(\overline{G}|\rho) P(\rho)}{P(\overline{G})} \propto P(\overline{G}|\rho) P(\rho) \]

Likelihood of fit parameters before doing the calculation.

\[ P(\rho) = e^{-\chi^2_{\text{prior}}(\rho)/2} \]

Likelihood of fit parameters given your data.

\[ P(\overline{G}|\rho) \propto e^{-\chi^2(\rho)/2} \]

Probability of your data given the fit parameters.
What do we know in advance about the fit function for form factors?

A nice ansatz, the Becirevic-Kaidalov parameterization:

\[
f_+(q^2) = \frac{F}{(1 - q^2)(1 - \alpha q^2)}, \quad f_0(q^2) = \frac{F}{1 - \frac{q^2}{\beta}}
\]

Correct effects of the B* pole that we know are there.

Additional pole parameterizes higher mass states.

Real life higher mass states for \(B \to \pi l \nu\): a cut.

\[
F_+(q^2) = \frac{F_+(0)/(1 - \alpha)}{1 - \frac{q^2}{m_{B^*}^2}} + \frac{1}{\pi} \int_{t_+}^\infty dt \frac{\text{Im} F_+(t)}{t - q^2 - i\epsilon}
\]

BK could be extended with increasing accuracy by adding more and more poles. (Hill.)

\[
F_+(q^2) = \frac{F_+(0)/(1 - \alpha)}{1 - \frac{q^2}{m_{B^*}^2}} + \sum_{k=1}^{N} \frac{\rho_k}{1 - \frac{1}{\gamma_k m_{B^*}^2}}
\]
Current limitations of BK

1) In lattice data fit with BK, statistical errors are smaller at high momentum than at low.

An effect of the model not present in the raw data.

2) Richard Hill: experimental data better fit with an extended BK with an extra parameter:

\[
F_+(q^2) = \frac{F_+(0)/(1-\alpha)}{1-q^2/m_{B*}^2} + \frac{c}{1-q^2/M'^2} + \cdots
\]

\[
= \frac{F_+(0)(1-\delta q^2/m_{B*}^2)}{(1-q^2/m_{B*}^2)(1-\alpha)[\alpha+\delta(1-\alpha)]q^2/m_{B*}^2}
\]
What do we know in advance about the fit function for form factors?

Not easy to fix prior uncertainties to parameters in BK extensions.

Analyticity and unitarity have long been used to constrain shapes of form factors.

A particularly simple form has recently been emphasized by Arnesen et al.

Consider a remapping of the semileptonic decay variable \( t = q^2 \) into a new variable \( z \) in the complex plane:

\[
z(t, t_0) = \frac{\sqrt{t_+ - t - \sqrt{t_+ - t_0}}}{\sqrt{t_+ - t + \sqrt{t_+ - t_0}}}
\]

where \( z \) maps \( q^2 = t > t_+ \) onto \( |z|=1 \), and \( t < t_+ \) onto \([-1, 1]\) in the complex plane.

\[
( t = (p_h-p_L)^2, \ t_+ = (m_H+m_L)^2, \ t = (m_H-m_L)^2 ).
\]

\( t_0 \), taken as 0.65 \( t \) here, is a fudge factor adjusted to center the physical region on \( z \approx 0 \).
A power series expansion of the form factors in $z$ can be written in the form:

$$f(t) = \frac{1}{P(t) \phi(t, t_0)} \sum_{k=0}^{\infty} a_k(t_0) z(t, t_0)^k$$

Function that has unit norm at $z=1$, and that vanishes at the poles of $f$, e.g., at the $B^*$ pole.

$P$ and $\phi$ contain most of the complexity of the form factors.
By calculating the current-current correlation function in perturbation theory and using the $J^{\mu}B\pi$ amplitude,

$$\text{Im} \Pi^{\mu\nu} = \int [\text{p.s.}] \ delta (q - p_{B\pi}) \langle 0 | J^{\dagger \nu} \bar{B}\pi \rangle \langle \bar{B}\pi | J^{\mu} | 0 \rangle + \ldots$$

with crossing symmetry and analyticity, one obtains a simple constraint on the $a_k$s in the equation

$$f(t) = \frac{1}{P(t) \phi(t, t_0)} \sum_{k=0}^{\infty} a_k(t_0) \ z(t, t_0)^k$$

It is simply

$$\sum_{k=0}^{n_A} a_k^2 \leq 1 \ !$$
The allowed range of $z$ in physical semileptonic decay is small

- $B \to \pi \ell \nu$: $-0.34 < z < 0.22$,
- $D \to \pi \ell \nu$: $-0.17 < z < 0.16$,
- $D \to K \ell \nu$: $-0.04 < z < 0.06$,
- $B \to D \ell \nu$: $-0.02 < z < 0.04$.

\[
f(t) = \frac{1}{P(t)\phi(t, t_0)} \sum_{k=0}^{\infty} a_k(t_0) z(t, t_0)^k
\]

Since \( \sum_{k=0}^{n_A} a_k^2 \leq 1 \),

to obtain the form factors to high accuracy, say 1\%, only a small number of parameters is needed, only 5 or 6 even in the case of $B \to \pi \ell \nu$. 

![Diagram](Image)
$B \rightarrow \pi l \nu$, unitarity fits

Strong $q^2$ dependence in form factor is due to calculable effects. When those are factored out, two parameters suffice to describe the current experimental data. (Just like $B \rightarrow D l \nu, K \rightarrow \pi l \nu$?!!)
$B \rightarrow \pi l\nu$, unitarity fits

$B \rightarrow \pi$ form factor data normalized by $P(t) \times \Phi(t, t_0)$ vs. $z(t)$

Coefficients in $z$ expansion are compatible with experiment.

$a_0$: 0.0257228 +- 0.003
$a_1$: 0.020256 +- 0.068
$a_2$: 0.152234 +- 0.41

<<1: see Becher, Hill.
$B \rightarrow \pi \nu$, unitarity fits

Combined fits of $f^+$ and $f_0$ may give surprisingly good prediction for form factors well beyond the range of lattice data.

- Raw lattice data,
- Not extrapolated in $m$ or $a$,
- Momentum dependent discretization errors not yet included.

How can the results of such fits best be compared with experiment?
Summary

- Fermilab and MILC are calculating an extensive set of semileptonic form factors at several lattice spacings.
  - \( D \rightarrow \{ \pi, K \} l \nu \)
  - \( B \rightarrow D\{^*\} l \nu \)
  - \( B \rightarrow \pi l \nu \)
  - \( B \rightarrow K l^+ l^- \)

- The analyticity-based \( z \) expansion limits the number of parameters needed to describe form factor data, without introducing model dependence.

- In terms of the \( z \) expansion, all semileptonic form factor data, both lattice and experiment are consistent with straight lines: normalization and slope.
  - Even \( B \rightarrow \pi l \nu \).