Pendulum with vibrating pivot: bifurcation diagram

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Theory

A pendulum with a vibrating pivot point is an example of a parametric resonance. It is similar to a chaotic pendulum, but with the driving force depending on sin θ [1]:

$$\frac{d^2\Theta}{dt^2} = -a\frac{d\Theta}{dt} - (w_0 + f\cos wt)\sin\Theta$$

One way of understanding the physics of this equation is to go to the rest frame of the pivot (an accelerating reference frame) where you would say that there is a fictitious force that effectively leads to a sinusoidal variation of g or w_0^2.

Analytic as well as numeric studies of this system exist. A fascinating aspect of this system is that the excitation of its modes of vibration occurs through a series of bifurcations. In fact, when the instantaneous angular velocity is plotted as a function of the strength of the driving force, the bifurcation diagram results. Although the physics is very different, this behavior is manifestly similar to the bifurcation diagram for bug populations. This behavior is, apparently, the result of mode locking and beating with modes of vibration [1].

Method

First, the above equation can be changed into a first order differential equation by expressing angular velocity as s. Then the above equation becomes:

$$\frac{ds}{dt} = -as - (w_0 + f\cos wt)\sin\Theta$$

Then I can use Euler's method to evolve the equation with time. To better approximation, I would use Runge-Kutta method for step-size increment as follows:

$$\Theta_{i+1} = \Theta_i + n$$

$$s_{i+1} = s_i + m$$

where

$$n = \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$$

Where

k_1 = si

$k_2 = s(t_i + 1/2h, \Theta_i + 1/2hk_1)$

$k_3 = s(t_i + 1/2h, \Theta_i + 1/2hk_2)$

$k_4 = s(t_i + h, \Theta_i + hk_3)$

the form of m is the same as n except Θ_i is replaced by $\frac{ds}{dt}$. h is the step-size in t.

Every time the driving force changes its sign, I will record absolute value of s. After a sufficient number of periods have passed, plotting absolute value of s versus amplitude of driving force, f, would produce bifurcation diagram. The graph can be plotted by generating data from the program and plot it through gnuplot.

Implementation

First of all, set alpha = .1, $w_0 = 1$, w = 2, and let f vary through the range. Also set the initial conditions $\theta(0) = \theta_0 = 1$ and s(0) = 1. Record the instantaneous angular velocity whenever the driving force passes through zero. One way to do this is record s whenever driving force changes its sign. Wait 150 periods before sampling to permit transients to die off. This means evolve the equation from t = 0 to t = 150/w. Sample angular velocity for 150 times and plot the results. Also plot absolute value of s vs. f. Plotting can be achieved using fstream data generation code. I set some values above but one can choose different values to see variations. But in general this would result in bifurcation diagram [2].

Reference

[1]. Classical Mechanics, John Taylor, University Science Books, ISBN-10: 9781891389221.

[2]. Computational Physics: Problem Solving with Computers, Rubin H. Landau, Manuel J. Paez, Wiley-interscience Publication, ISBN-0-471-11590-8.

(I have not included detailed codes in this proposal. I am still having problems with writing codes for this example. I'd like to discuss the issue with you and I can re-write a better proposal later)