Probing New Physics in $B \to D^* \ell \bar{\nu}_\ell$ decays Using a Monte-Carlo Event Generator

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Outline

1. Motivation
   - for a MC Event Generator
   - for studying the $B \to D^* \ell \bar{\nu}_\ell$ decays.
2. Theory Formalism
3. Observables and potential impact of general NP operator structures.
4. Summary & Outlook

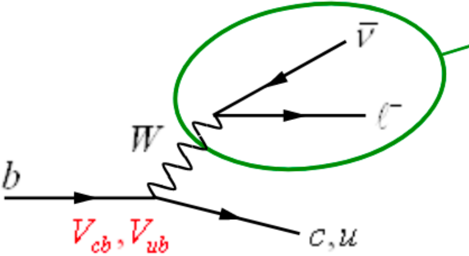
Motivation

Monte-Carlo Event Generator : EvtGen

- A program which simulates particle physics events with the same probability as they occur in nature.
- Hence, they help us to give accurate theoretical predictions and devise experimental strategies.
- **EvtGen** : is a MC event generator that simulates the decays of heavy flavour particles, primarily the B and D mesons.
  - Originally written by Anders Ryd and David Lange.
  - It has detailed models for semileptonic decays, CP-violating decays and produces correct results for the angular distributions in sequential decays, including all correlations.
  - At the moment only the SM is implemented.
  - We have developed the **NP module** for the $B \to D^* \ell \bar{\nu}_\ell$ decay https://github.com/qdcampagna/BTODSTARLNUNP˙EVTGEN˙Model.

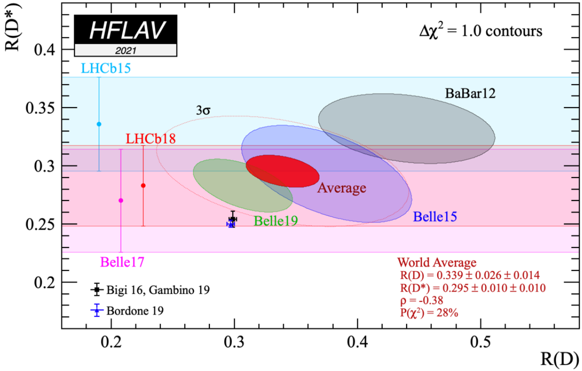
Why $B \to D^* \ell \bar{\nu}_\ell$?

- Semileptonic decays are theoretically clean : Leptonic current is decoupled from the hadronic current.
- Useful in the extraction of $|V_{cb}|$.
- Testing CKM unitarity.
- Sensitive probes of New Physics.
- Test Lepton Flavour Universality of the SM.
- Understanding CP violation in B-meson system.



Current Experimental Status

$$R_{D^{(*)}} = \frac{\mathcal{B}(\bar{B}\to D^{(*)}\tau\bar{\nu}_\tau)}{\mathcal{B}(\bar{B}\to D^{(*)}\ell\bar{\nu}_\ell)} \text{ (with } \ell = e \text{ or } \mu)$$



- $R_D^{SM} = 0.298 \pm 0.003$, $R_{D^*}^{SM} = 0.252 \pm 0.005$
- The WA corresponds to a combined 3.4σ deviation from the SM.

Possible new physics in tau!

Current Experimental Status

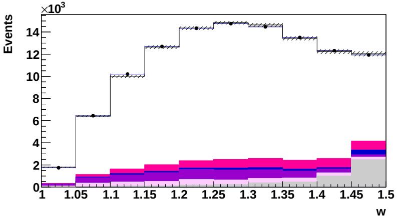


Figure: Signal (black) and background (colored) events for electron (left) and muon (right) modes for the $B^0 \to D^{*-}\ell^+\nu_\ell$ decays, reconstructed with the full Belle data set of 711 fb^{-1} integrated luminosity *(Phys. Rev. D 100 (2019), 052007)*.

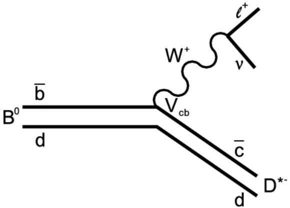
$$w = \frac{m_B^2 + m_{D^*}^2 - q^2}{2 m_B m_{D^*}}$$

- The electron and muon data are in good agreement with SM!
- Hence, these modes were always used in the extraction of V_{cb}.
- But, there have been recent hints of NP in muon as well, so stay tuned!

Theory Formalism

Effective Field Theory

- An EFT is useful when a dynamical process involves many energy scales.



$$M_W >> m_b \;\&\; M_W >> q^2$$

- An effective theory is formulated based only on low-energy assumptions, i.e. by specifying the dynamical fields and symmetries etc.
- It helps us to separate out the energy scales involved in the Lagrangian.

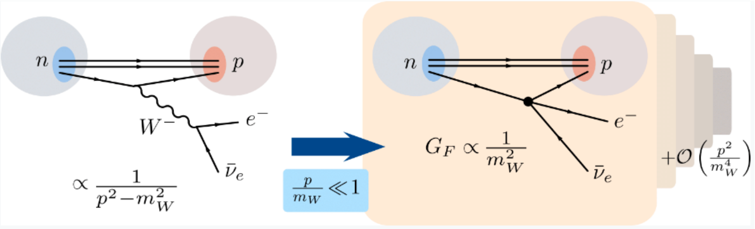
$$\mathcal{L} = \mathcal{L}_0 + \frac{\mathcal{L}_1}{\Lambda} + \frac{\mathcal{L}_2}{\Lambda^2} + ... + \frac{\mathcal{L}_n}{\Lambda^n}$$

where $\mathcal{L}_0$ is at the energy scale of your process/experiment (say Λ_0) and Λ is a high energy scale $>> \Lambda_0$ (could be the new physics scale).

- The separation of scales makes the theory simpler by removing irrelavant degrees of freedom and extracting useful information of physical processes that are relevant at that low scale.

Effective Field Theory

Fermi's theory of beta decays is by far the best example of an EFT!



$$
\begin{aligned}
\mathcal{M} &\propto (\bar{\psi}_p \gamma^\mu \psi_n) \frac{1}{q^2 - M_W^2} (\bar{\psi}_e \gamma_\mu \psi_\nu) \\
&\propto -\frac{1}{M_W^2} (\bar{\psi}_p \gamma^\mu \psi_n) \left(1 - \frac{q^2}{M_W^2}\right)^{-1} (\bar{\psi}_e \gamma_\mu \psi_\nu) \\
&= -\frac{G_F}{\sqrt{2}} (\bar{\psi}_p \gamma^\mu \psi_n)(\bar{\psi}_e \gamma_\mu \psi_\nu) + \mathcal{O}(1/M_W^4) + \ldots
\end{aligned}
$$

Fermi's Constant : $G_F = \frac{\sqrt{2}}{8} \frac{g^2}{M_W^2}$

Effective Hamiltonian for $b \to c\ell\bar{\nu}$ decays

$$
\mathcal{H}_{eff} = \frac{G_F V_{cb}}{\sqrt{2}} \Big\{ (1 + g_L) \left[\bar{c}\gamma_\mu(1-\gamma_5)b\right] \left[\bar{\ell}\gamma^\mu(1-\gamma_5)\nu_\ell\right]
+ g_R \left[\bar{c}\gamma_\mu(1+\gamma_5)b\right] \left[\bar{\ell}\gamma^\mu(1-\gamma_5)\nu_\ell\right]
+ g_S \left[\bar{c}b\right] \left[\bar{\ell}(1-\gamma_5)\nu_\ell\right]
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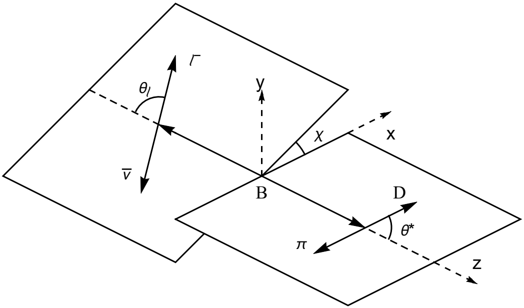
4. Hadronic matrix elements are expressed in terms of form factors which are non-perturbative objects (cannot be calculated form first principles of QCD).

Kinematics

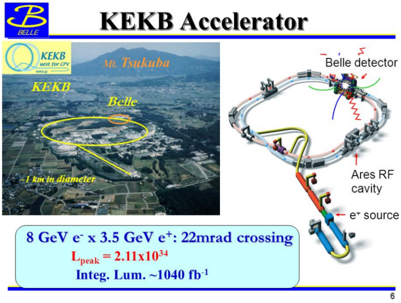


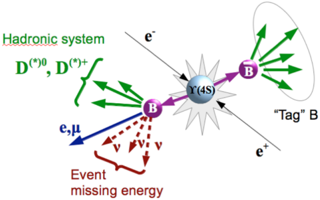
- q^2 : the lepton-neutrino invariant mass squared.
- θ_ℓ : the angle between the direction of the lepton & the direction opposite the D^* meson in the virtual W rest frame.
- θ_{D^*} : the angle between the direction of the D^0 meson & the direction of the D^* meson in the D^* rest frame.
- χ : azimuthal angle between the two decay planes.

Observables & New Physics Analysis

NP Analysis with our MC Generator

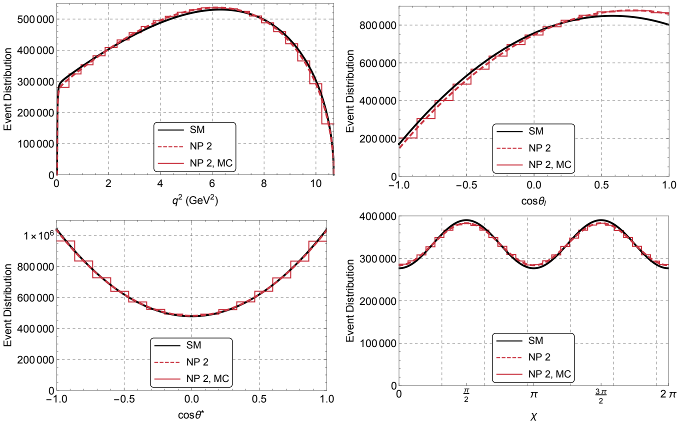
- Using our MC we simulate 10^7 events (anticipated Belle II statistics at 50 ab^{-1}).
  - KEKB is an asymmetric e^+e^- collider located in Tsukuba, Japan.
  - $B\bar{B}$ pairs are produced from decay of $\Upsilon(4S)$.
  - Belle II is the high luminosity upgrade of the Belle detector (projected luminosity reach of 50 ab^{-1}).
  - Ideal for the study of the $B \to D^* \ell \nu$ decays.
- Throughout our analysis, NP is considered ONLY in the muon final states.





Courtesy : J. Phys.: Conf. Ser. 1271 012011

NP Sensitivities of Decay Distributions



MC Simulated for NP : $g_L = 0.08, g_R = 0.09, g_P = 0.6i$

Least sensitive to NP!

Observable : Angular Asymmetries

Angular observables are more sensitive to NP!

$$\frac{d^2\Gamma}{dq^2 d\cos\theta^*} = \frac{3}{4}\frac{d\Gamma}{dq^2}\left[2\,F_L^{D^*}(q^2)\cos^2\theta^* + F_T^{D^*}(q^2)\sin^2\theta^*\right],$$

$$\frac{d^2\Gamma}{dq^2 d\cos\theta_\ell} = \frac{d\Gamma}{dq^2}\left(\frac{1}{2} + A_{FB}\cos\theta_\ell + \frac{1-3\,\tilde{F}_L^\ell}{4}\,\frac{3\cos^2\theta_\ell - 1}{2}\right),$$

$$\frac{d^2\Gamma}{dq^2 d\cos\chi} = \frac{1}{2\pi}\frac{d\Gamma}{dq^2}\left(1 + S_3\cos 2\chi + S_9\sin 2\chi\right)$$

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- Longitudinal Polarisation of D^*
- Forward-backward Asymmetry
- Triple-product Asymmetry
- ... etc.

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Our main observables of interest : $\mathbf{A_{FB}, S_3, S_5, S_7}$ where S_5, S_7 are constructed from the distribution by performing asymmetric integrals over more than one angles.

Forward-Backward Asymmetry A_{FB}

- Asymmetries help us to distinguish between particles and anti-particles.
- In general, the forward backward asymmetry is defined as :

$$\mathcal{A}_{FB} = \frac{N_F - N_B}{N_F + N_B}$$

Number of fermions produced in forward region (N_F)

Number of fermions produced in backward region (N_B)

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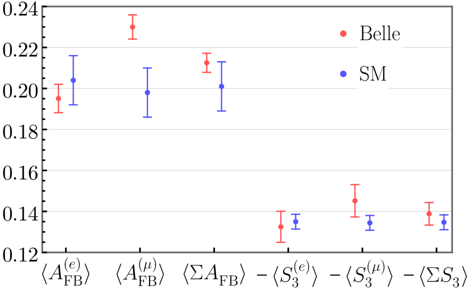
$$A_{FB}(q^2) = \left(\frac{d\Gamma}{dq^2}\right)^{-1} \left[\int_0^1 - \int_{-1}^0\right] d\cos\theta_\ell \frac{d^2\Gamma}{d\cos\theta_\ell dq^2} \quad (1)$$

$\cos\theta_\ell > 0$: Forward Region

$\cos\theta_\ell < 0$: Backward Region

Forward-Backward Asymmetry A_{FB}

Recently, ***Bobeth et al.*** *(Eur. Phys. J. C **81** (2021) no.11, 984)*, studied the CP-averaged lepton forward-backward asymmetry $\langle \mathcal{A}_{FB}^{\ell} \rangle$ for $\ell = e, \mu$ based on the binned differential rates provided by ***Belle*** *(Phys. Rev. D 100 (2019), 052007)*



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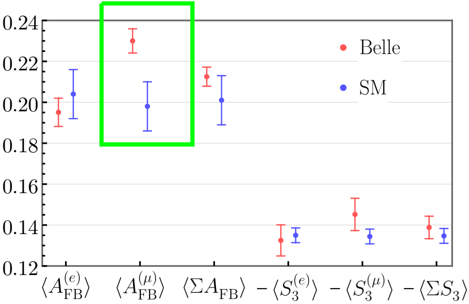

$\langle A_{FB}^{\mu} \rangle$ shows a tension above the 2σ level.

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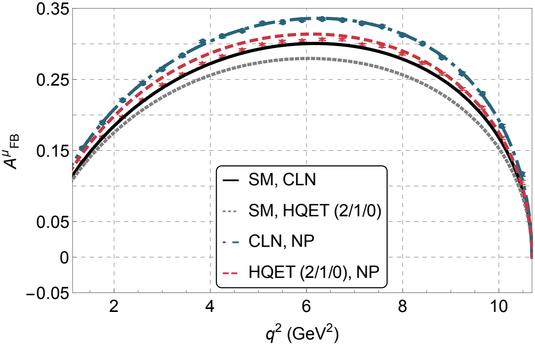

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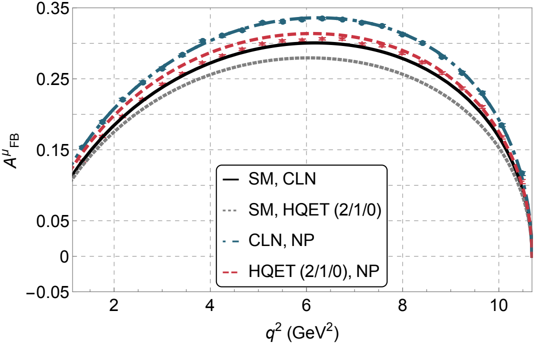

- However, it is seen that "Δ"-observables : $\Delta X = X^{\mu} - X^{e}$ have reduced sensitivity to form factor uncertainties.

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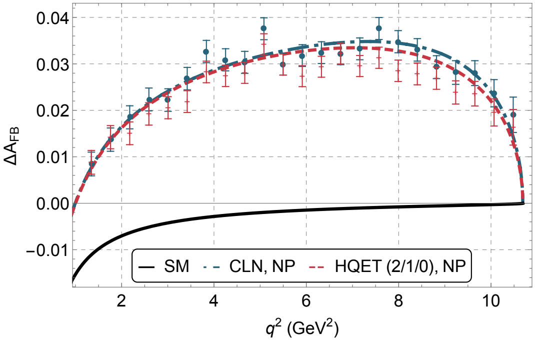

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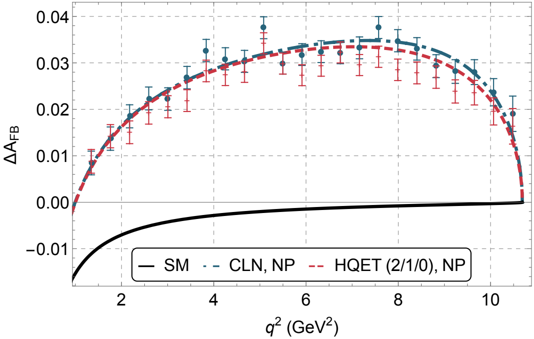


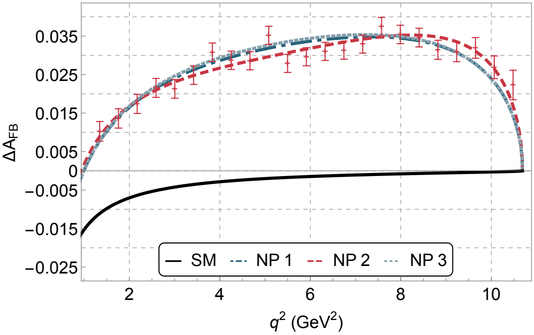
- So any deviation in ΔA_{FB} from the SM expectation is an unambiguous signal of NP!

NP Scenarios

- We pick out a few NP scenarios as listed below.
- The choice is motivated such that :
  - the ratio of semi-leptonic branching fractions is constrained to be within 3% of unity.
  - they are able to explain the experimental $\langle \Delta A_{FB} \rangle$ i.e within 0.0349 ± 0.0089.
  - they also satisfy constraints on other angular observables such as $\langle \Delta F_L \rangle^{exp} = -0.0065 \pm 0.0059$ and $\langle \Delta \tilde{F}_L \rangle^{exp} = -0.0107 \pm 0.0142$.

	g_L	g_R	g_P
Scenario 1:	0.06	0.075	0.2 i
Scenario 2:	0.08	0.090	0.6 i
Scenario 3:	0.07	0.075	0

NP Analysis of Angular Asymmetries



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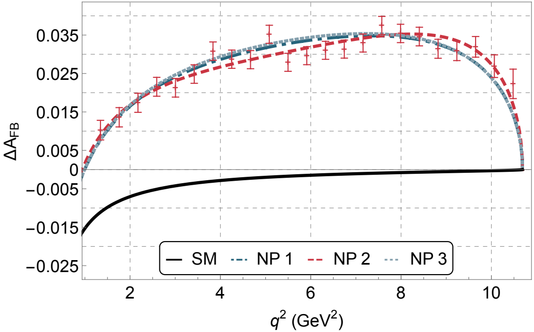
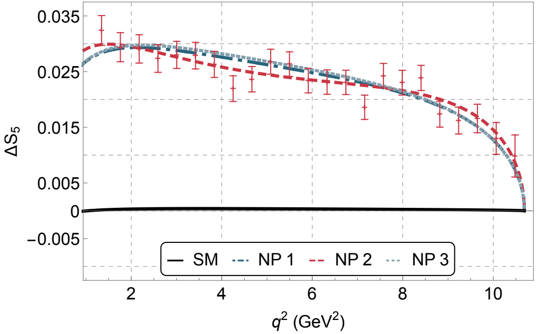


- We find that the NP scenarios produce correlated signatures of deviation in ΔA_{FB} and ΔS_5.

NP Analysis of Angular Asymmetries

Double-angle asymmetry :

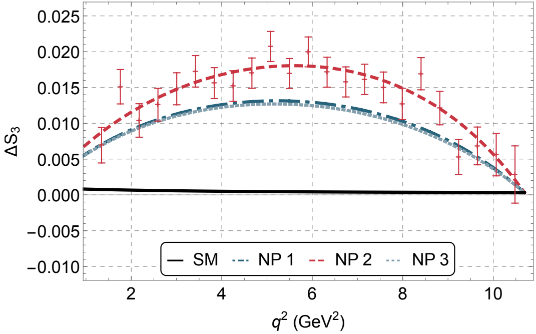
$$S_5(q^2) = \left(\frac{d\Gamma}{dq^2}\right)^{-1} \left[\int_0^{\pi/2} - \int_{\pi/2}^{\pi} - \int_{\pi}^{3\pi/2} + \int_{3\pi/2}^{2\pi}\right] d\chi \left[\int_0^1 - \int_{-1}^0\right] d\cos\theta^* \frac{d^3\Gamma}{dq^2\, d\cos\theta^*\, d\chi} \quad (2)$$



NP Analysis of Angular Asymmetries

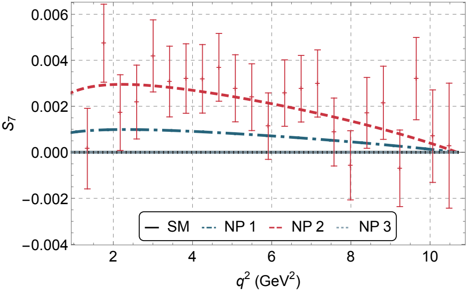
χ-angle asymmetry :

$$S_3(q^2) = \left(\frac{d\Gamma}{dq^2}\right)^{-1} \left[\int_0^{\pi/4} - \int_{\pi/4}^{\pi/2} - \int_{\pi/2}^{3\pi/4} + \int_{3\pi/4}^{\pi} + \int_{\pi}^{5\pi/4} - \int_{5\pi/4}^{3\pi/2} - \int_{3\pi/2}^{7\pi/4} + \int_{7\pi/4}^{2\pi}\right] d\chi \frac{d^2\Gamma}{dq^2 d\chi} \quad (3)$$



NP Analysis of Angular Asymmetries

$$S_7(q^2) = \left(\frac{d\Gamma}{dq^2}\right)^{-1} \left[\int_0^{\pi} - \int_{\pi}^{2\pi}\right] d\chi \left[\int_0^{1} - \int_{-1}^{0}\right] d\cos\theta^* \, \frac{d^3\Gamma}{dq^2 \, d\cos\theta^* \, d\chi} \qquad (4)$$



- This is a true CP-odd triple-product asymmetry, which is predicted to be identically zero in the SM for any q^2.
- NP scenarios with an imaginary g_P are able to produce a small non-zero signal in the q^2 distribution of S_7.
- This observable tells us whether there is any complex NP.

Summary & Outlook

Summary & Outlook

- We have successfully developed the NP MC generator for $B \to D^* \ell \nu$ decays.
- Work on tensor type NP is still in progress.
- We also plan to perform detailed experimental sensitivity studies using full Belle II detector MC simulation, backgrounds, detector efficiencies etc.
- Study the tau mode which has provided the strongest hints of NP in this decay channel so far.

THANK YOU!

Backup Slides

Backup

$$\frac{d^4\Gamma}{dq^2\, d\cos\theta^*\, d\cos\theta_\ell\, d\chi} = \frac{9}{32\pi}\Big[\big(I_1^s \sin^2\theta^* + I_1^c \cos^2\theta^*\big)$$
$$+ \big(I_2^s \sin^2\theta^* + I_2^c \cos^2\theta^*\big)\cos 2\theta_\ell + I_3 \sin^2\theta^* \sin^2\theta_\ell \cos 2\chi +$$
$$I_4 \sin 2\theta^* \sin 2\theta_\ell \cos\chi + I_5 \sin 2\theta^* \sin\theta_\ell \cos\chi$$
$$+ \big(I_6^c \cos^2\theta^* + I_6^s \sin^2\theta^*\big)\cos\theta_\ell + I_7 \sin 2\theta^* \sin\theta_\ell \sin\chi$$
$$+ I_8 \sin 2\theta^* \sin 2\theta_\ell \sin\chi + I_9 \sin^2\theta^* \sin^2\theta_\ell \sin 2\chi\Big]$$